Warwick Mathematics Exchange

MA3F1

Introduction to Topology

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Desync, aka The Big Ree

Contents

Introduction

Topology is the branch of mathematics concerned with continuity and connectedness, and properties of spaces that are invariant under continuous deformations.

For instance, the *hairy ball theorem* of algebraic topology states that there is no nonvanishing continuous tangent vector field on the sphere (or "you can't comb the hair on a coconut flat"). The object in question being a sphere is not actually important to the theorem, and it holds on any smooth blob that can be continuously deformed into a sphere. Note that this excludes say, a torus, which has a hole and thus cannot be continuously deformed into a sphere. Topology makes precise this distinction between a sphere and a torus ("homotopy classes"), and also formalises what it means to continously deform an object ("homeomorphisms").

Disclaimer: I make *absolutely no guarantee* that this document is complete nor without error. In particular, any content covered exclusively in lectures (if any) will not be recorded here. This document was written during the 2023 academic year, so any changes in the course since then may not be accurately reflected.

Notes on formatting

New terminology will be introduced in *italics* when used for the first time. Named theorems will also be introduced in *italics*. Important points will be **bold**. Common mistakes will be <u>underlined</u>. The latter two classifications are under my interpretation. YMMV.

Content not taught in the course will be outlined in the margins like this. Anything outlined like this is not examinable, but has been included as it may be helpful to know alternative methods to solve problems.

The table of contents above, and any inline references are all hyperlinked for your convenience.

History

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Please send me a PM on Discord @Desync#6290, a message in the WMX server, or an email to Warwick.Mathematics.Exchange@gmail.com for any corrections. (If this document somehow manages to persist for more than a few years, these contact details might be out of date, depending on the maintainers. Please check the most recently updated version you can find.)

If you found this guide helpful and want to support me, you can buy me a coffee!

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*Storing dates in big-endian format is clearly the superior option, as sorting dates lexicographically will also sort dates chronologically, which is a property that little and middle-endian date formats do not share. See ISO-8601 for more details. This footnote was made by the computer science gang.

1 Glossary

isomorphism, $\cong$	A morphism with a two-sided inverse. Does not necessarily correspond with a bijective morphism in any given category.
homeomorphism, $\cong$	A bicontinuous bijection of topological spaces; an isomorphism in **Top**.
quotient map	The function $q : X \to X/\sim$ canonically defined by $x \mapsto [x]_\sim$.
identification map	A continuous surjection that preserves openness of sets in both directions.
cover	A collection of (open) sets whose union is the covered space.
Lebesgue number	Given a cover $\mathcal{U}$ of a metric space (X,d), a number $\delta > 0$ is a Lebesgue number for $\mathcal{U}$ if for every $x \in X$ there exists an open set $U \in \mathcal{U}$ such that $\mathbb{B}(x,\delta) \subseteq U$. Or equivalently, $\delta > 0$ is a Lebesgue number for $\mathcal{U}$ if every subset $S \subseteq X$ with diameter at most $\operatorname{diam}(S) \leq \delta$ is contained within some member of the cover.
retract	A subset $A \subseteq X$ is a retract of X if there is a continuous map $r : X \to A$ such that $r\big\vert_A = \mathrm{id}_A$, called the retraction.
(strong) deformation retract	A subset $A \subseteq X$ is a (strong) deformation retract of X if there exists a one-parameter family of maps $f_t : X \to X$, $t \in I$ (or by uncurrying, a single map $F : X \times I \to X$), such that $f_0 = \mathrm{id}_X$; $f_1(X) = A$; and $f_t\big\vert_A = \mathrm{id}_A$ for all $t \in I$.
weak deformation retract	Same as the above, but the final condition is relaxed to only $t = 1$.
(free) homotopy, $f \simeq g$	A continuous map $F : X \times I \to Y$ is a homotopy between the maps f_1 and f_2.
relative homotopy, $f \overset{A}{\simeq} g$	A homotopy that additionally fixes some subspace $A \subseteq X$ for all $t \in I$.
homotopy relative to the boundary, $f \overset{\partial}{\simeq} g$	A relative homotopy where A is the pair of endpoints of the paths.
linear homotopy	The homotopy between f and g given by $x \mapsto (1-t)f(x) + tg(x)$.
homotopy equivalence, $X \simeq Y$	A relaxation of isomorphism that only requires that the composition is homotopic and not equal to the identity.

contractible	A space is contractible if it is homotopy equivalent to the one-point space.
null-homotopic	A map is null-homotopic if it is homotopic to some constant map.
connected	A space is connected if it cannot be partitioned into two disjoint open sets.
path-connected	A space is path-connected if every pair of points may be connected by a path.
neighbourhood basis	A neighbourhood basis at a point x is a collection $\mathcal{B}$ of neighbourhoods of x such that for any neighbourhood V of x, there exists a neighbourhood $B \in \mathcal{B}$ such that $B \subseteq V$.
locally connected	A space is locally connected if every point admits a neighbourhood basis consisting of connected sets.
locally path-connected	A space is locally path-connected if every point admits a neighbourhood basis consisting of path-connected sets.
cover	A covering of a space X is a map $p : \tilde{X} \to X$ such that for every point $x \in X$, there exists an open neighbourhood $U_x \subseteq X$ of x and a discrete (i.e. every set is open) space $D_x \subseteq \tilde{X}$ such that $p^{-1}[U_x] = \bigsqcup_{d \in D_x} V_d$ and the restriction $p\big\|_{V_d} : V_d \to U_x$ is a homeomorphism for every $d \in D_x$. The sets U_x are said to be evenly covered, and the open sets V_d are called the sheets of the covering. The pair $(\tilde{X},p)$ is then a cover of X.
fibre	The preimage of a singleton set under a covering.
deck transformation	Given a covering $p : \tilde{X} \to X$, a deck transformation is a homeomorphism $\tau : \tilde{X} \to \tilde{X}$ such that $p \circ \tau = p$.
lift, $\tilde{f}$	Given a covering $p : \tilde{X} \to X$ and a map $f : Y \to X$, a lift of f is a map $\tilde{f} : Y \to \tilde{X}$ such that $p \circ \tilde{f} = f$.
induced homomorphism, f_*	Given a pointed map $f : (X,x_0) \to (Y,y_0)$, the induced homomorphism of fundamental groups is the homomorphism $f_* : \pi_1(X,x_0) \to \pi_1(Y,y_0)$ defined by $[\alpha] \mapsto [f \circ \alpha]$.
odd and even function	A map $f : X \to Y$ is odd if $f(-x) = -f(x)$; and even if $f(-x) = f(x)$; for all $x \in X$.

wedge sum, $\vee$	The one point union of a collection of spaces; the disjoint union of a collection of pointed spaces with the basepoints identified. This identified point is a natural basepoint for the wedge sum, and picking this point makes the wedge sum associative and commutative (up to homeomorphism).
reduced word	A word in a free product is reduced if it does not contain any identities, and if every pair of consecutive letters is not from the same group.

2 Review of Point-Set Topology

2.1 Metric Spaces

Let X be any set. A *metric* d on X is a map $d : X \times X \to \mathbb{R}_{\geq 0}$ such that,

(i) $d(x,y) = 0$ if and only if $x = y$ (point separating or positive-definiteness);

(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$ (symmetry);

(iii) $d(a,b) \leq d(a,x) + d(x,b)$ for every $a,b,x \in X$ (triangle inequality).

Note that these axioms imply that $d(x,y) \geq 0$ for all $x,y \in X$. The pair (X,d) is then called a *metric space.*

Let (X,d) be a metric space. The *open ball* centred at $a \in X$ of radius r is the set

$$\mathbb{B}(a,r) = \{x \in X : d(x,a) < r\}$$

also denoted by $B(a,r)$ or $\mathbb{B}_r(a)$. If $r = 1$, we say that the ball is a *unit ball*, and we omit r from the notation.

In a metric space (X,d), a set $U \subseteq X$ is said to be *open in* X if for every point $x \in U$, there exists some $\varepsilon > 0$ such that $\mathbb{B}(x,\varepsilon) \subset U$. A set $U \subseteq X$ is said to be *closed in* X if its complement is open in X. If the ambient set X is clear, then we omit the "in X" and just say that a set is open or closed.

Example.

- In any metric space (X,d), X and $\emptyset$ are both simultaneously open and closed (or *clopen*).
- In $\mathbb{R}$, open intervals are open and closed intervals are closed. Half-open intervals are neither open nor closed.
- In a discrete metric space, every singleton set $\{x\} \subseteq X$ is open (take any $\varepsilon < 1$).

Sets can be open, closed, both (clopen), or neither, so the adjectives "open" and "closed" do not have all of their usual intuitive connotations when used in a mathematical context.

Lemma (Open Finite Intersection)**.** *If $(U_i)_{i=1}^n$ is a finite collection of sets open in (X,d), then $\bigcap_{i=1}^n U_i$ is open in (X,d).*

Proof. Take $x \in \bigcap_{i=1}^n U_i$. Then, for each i, $x \in U_i$, so there exists $\varepsilon_i > 0$ such that $\mathbb{B}_{\varepsilon_i}(x) \subset U_i$. If $\varepsilon := \min(\varepsilon_1, \ldots, \varepsilon_n)$, then,

$$\mathbb{B}_\varepsilon(x) \subseteq \mathbb{B}_{\varepsilon_i}(x) \subset U_i$$

for all i, and hence $\mathbb{B}_\varepsilon(x) \subset \bigcap_{i=1}^n U_i$. ■

Lemma (Open Arbitrary Union)**.** *If $(U_i)_{i\in\mathcal{I}}$ is an arbitrary collection of sets open in (X,d), then $\bigcup_{i\in\mathcal{I}} U_i$ is open in (X,d).*

Proof. If $x \in \bigcup_{i\in\mathcal{I}} U_i$, then $x \in U_i$ for some $i \in \mathcal{I}$. Since U_i is open, there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \subset U_i \subseteq \bigcup_{i\in\mathcal{I}} U_i$, so $\bigcup_{i\in\mathcal{I}} U_i$ is open. ■

By De Morgan's laws, we also have:

Corollary (Closed Finite Union)**.** *If $(F_i)_{i=1}^n$ is a finite collection of sets closed in (X,d), then $\bigcup_{i=1}^n F_i$ is closed in (X,d).*

Proof.

$$X \setminus \bigcup_{i=1}^{n} F_i = \bigcap_{i=1}^{n} (X \setminus F_i)$$

As F_i is closed, $X \setminus F_i$ is open, so $\bigcap_{i=1}^{n}(X \setminus F_i)$ is the finite intersection of open sets, and hence $X \setminus \bigcup_{i=1}^{n} F_i$ is open. It follows that $\bigcup_{i=1}^{n} F_i$ is closed. ∎

Corollary (Closed Arbitrary Intersection)**.** *If $(F_i)_{i \in \mathcal{I}}$ is an arbitrary collection of sets closed in (X,d), then $\bigcup_{i \in \mathcal{I}} F_i$ is closed in (X,d).*

Proof.

$$X \setminus \bigcap_{i \in \mathcal{I}} F_i = \bigcup_{i \in \mathcal{I}} (X \setminus F_i)$$

As F_i is closed, $X \setminus F_i$ is open, so $\bigcup_{i \in \mathcal{I}}(X \setminus F_i)$ is the intersection of open sets, and hence $X \setminus \bigcap_{i \in \mathcal{I}} F_i$ is open. It follows that $\bigcap_{i \in \mathcal{I}} F_i$ is closed. ∎

2.2 Topological Spaces

Many properties of a metric space do not depend on our exact choice of metric, and many familiar notions such as convergence and continuity may be defined in terms of open sets, with no mention of a metric at all. This motivates the introduction of a more general kind of space defined entirely in terms of open sets.

A *topology* on a set X is a collection Ω of subsets of X, such that

(T1) X and $\emptyset$ are open;

(T2) If $(U_i)_{i \in \mathcal{I}} \subseteq \Omega$, then $\bigcup_{i \in \mathcal{I}} U_i \in \Omega$ (arbitrary unions of open sets are open);

(T3) If $U, V \in \Omega$, then $U \cap V \in \Omega$ (binary intersections of open sets are open).

The pair (X,Ω) is then a *topological space*. We call the sets in Ω "open". Note that by induction, (T3) implies that the finite intersection of open sets is open.

These axioms mimic the ways open sets in metric spaces behave, but without reference to any kind of metric. Every metric space induces a topological space; and conversely, if a topology is induced by some metric, then the topology is said to be *metrisable*.

We often omit the topology from notation and speak about a set X as a topological space alone. Additionally, unless otherwise stated, when considering a metric space (X,d) as a topological space, the topology used will always be the topology induced from the metric.

The *closed* sets in a topological space are the complements of open sets. By De Morgan's laws, the collection $\mathcal{F}$ of closed sets satisfies:

(T1') T and $\emptyset$ are closed;

(T2') Arbitrary intersections of closed sets are closed.

(T3') The union of finitely many closed sets is closed;

Let (X,Ω) be a topological space. A set $\mathcal{B} \subseteq \Omega$ is a *basis* for the topology Ω, or that $\mathcal{B}$ *generates* the topology Ω, if every open set can be written as the union of sets in $\mathcal{B}$. That is, for each $U \in \Omega$, there exists a collection $\{B_i\}_{i\in\mathcal{I}} \subseteq \mathcal{B}$ such that $\bigcup_{i\in\mathcal{I}} B_i = U$.

Given $x \in X$, a set $\mathcal{B} \subseteq \Omega$ is a *neighbourhood basis* for x, if for every open set U containing x there is a set in the basis containing x that is a subset of U. That is, if for every $U \in \Omega$ with $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

2.3 Maps and Topological Equivalence

Let X,Y be topological spaces. A function $f : X \to Y$ is *continuous* if for any open set $U \subseteq X$, the preimage $f^{-1}[U] = \{x \in X : f(x) \in U\}$ is open. Continuous functions are sometimes abbreviated to *maps.*

Lemma 2.1 (Pasting Lemma)**.** *Let $X = A \cup B$, with A,B both closed or both open in X, and let $f : X \to Y$ be a function such that the restrictions $f\big|_A$ and $f\big|_B$ are continuous. Then, f is continuous.*

Proof. We prove the case for open A,B.

Let $U \subseteq Y$ be open in Y. Then, $f^{-1}[U] = f\big|_A^{-1}[U] \cup f\big|_B^{-1}[U]$, and because $f\big|_A$ and $f\big|_B$ are continuous, $f\big|_A^{-1}[U]$ and $f\big|_B^{-1}[U]$ are open in A and B, respectively. Because A and B are open, $f\big|_A^{-1}[U]$ and $f\big|_B^{-1}[U]$ are also open in X, so $f^{-1}[U]$ is open in X as it is the union of open sets, and hence f is continuous.

Exchanging "open" with "closed" in the previous yields a completely analogous proof for closed A,B. ■

Given topological spaces X and Y, a continuous map $f : X \to Y$ is a (topological) isomorphism or a *homeomorphism* if there exists a continuous map $g : Y \to X$ such that

$$f \circ g = \mathrm{id}_Y \,, \qquad g \circ f = \mathrm{id}_X$$

If a homeomorphism between X and Y exists, then X and Y are isomorphic topological spaces, or are *homeomorphic*, and we denote this relation (as usual) as $X \cong Y$.

2.4 The Subspace Topology

Let (X,Ω) be a topological space, and $S \subseteq X$ be a subset. The *subspace topology* on S is the set

$$\Omega_S = \{U \cap S : U \in \Omega\} \tag{1}$$

and we call (S,Ω_S) a *subspace* of (X,Ω).

Example.

- The (unit) *n-sphere* $\mathbb{S}^n$ or S^n is a subspace of $\mathbb{R}^{n+1}$ defined by

$$S^n = \left\{ x \in \mathbb{R}^{n+1} : \|x\|_2 = \sum_{i=1}^{n+1} x_i^2 = 1 \right\}$$

  Note that the superscript denotes the dimension of the sphere, and not the ambient space it is contained within.

- The (closed, unit) *n-disc* $\mathbb{D}^n$ or D^n is a subspace of $\mathbb{R}^n$ defined by

$$D^n = \left\{ x \in \mathbb{R}^n : \|x\|_2 = \sum_{i=1}^{n+1} x_i^2 \leq 1 \right\}$$

  The unit disc is a special case of a closed ball centred at the origin with radius 1.

2.5 Product Spaces

Let X,Y be topological spaces. The *product topology** on $X \times Y$ is the topology generated by sets of the form $U \times V$ with U and V open in X and Y, respectively.

Example.

- The product topology on $\prod_{i=1}^{n} \mathbb{R}$ coincides with the Euclidean topology on $\mathbb{R}^n$ (the topology induced by the ℓ^2 metric).
- The *topological torus* $\mathbb{T}^n$ or T^n is defined as the n-fold product of 1-spheres:

$$T^n = \prod_{i=1}^{n} S^1$$

Unless otherwise qualified, "torus" usually refers to T^2 – the surface of a doughnut.

2.6 Disjoint Unions

Given a family of sets $\{X_i\}_{i\in\mathcal{I}}$, the *disjoint union* of this family is the set

$$\bigsqcup_{i\in\mathcal{I}} X_i = \bigcup_{i\in\mathcal{I}} \{(x,i) : x \in X_i\}$$

Each set in the disjoint union is forced to be disjoint from every other via the use of the auxilliary index i, marking which set each element came from, so taking a disjoint union cannot lose information like a union. Intuitively, each of the sets X_i is canonically isomorphic to the set $\tilde{X}_i = X_i \times \{i\}$, so each set is equipped with a canonical embedding into the disjoint union, and furthermore, the images of these embeddings partition the disjoint union.

Given two topological spaces X and Y, we can endow the disjoint union $X \sqcup Y$ of the underlying sets with a topology generated by the basis consisting of sets of the form $U \times \{i\}$ for some $i \in \mathcal{I}$ and $U \subseteq X_i$ open.

Intuitively, in the disjoint union, the component spaces are now considered to be part of a single new space, but each space is completely detached and isolated from every other space, and retains its original local topology.

2.7 The Quotient Topology

Recall that an equivalence relation $\sim$ on a set X is a relation such that for all $x,y,z \in X$,

- $x \sim x$ (reflexivity);
- if $x \sim y$, then $y \sim x$ (symmetry);
- if $x \sim y$ and $y \sim z$, then $x \sim z$ (transitivity).

The *equivalence class* $[x]$ of an element $x \in X$ under an equivalence relation $\sim$ is the set of all elements of X equivalent to x. That is, the set

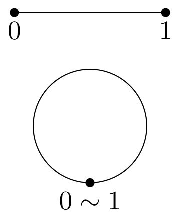
$$[x] = \{y \in X : x \sim y\}$$

The set of equivalence classes of an equivalence relation is denoted by $X/\sim$ and read as "the quotient of X by $\sim$", and the *quotient map* is the function $q : X \to X/\sim$ defined by $x \mapsto [x]$.

*More properly, this is the *box topology*, and not the true product topology, which is defined to be the coarsest topology such that the projections onto each component are all continuous. For finite product spaces, these topologies coincide, but for infinite products, the box topology is too fine and fails to satisfy a universal property.

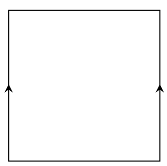
If X is a topological space, then the *quotient topology* on the set $X/\sim$ is defined to have a set $U \subseteq X/\sim$ open if and only if $q^{-1}[U] = \{x \in X : q(x) = [x] \in U\}$ is open in X. Note that by definition, the quotient map is continuous.

Example. Consider the unit interval $I = [0,1]$, and let $x \sim y$ if and only if $x = y$ and $0 \sim 1$ and $1 \sim 0$. The quotient set $I/\sim$, sometimes written as $I/0 \sim 1$ as only 0 and 1 are identified, then consists of the classes $[x] = \{x\}$ for $x \in (0,1)$ and $[0] = [1] = \{0,1\}$, so the endpoints of the interval have been "glued together" into a circle, and in fact, the resulting space with the quotient topology is homeomorphic to S^1.

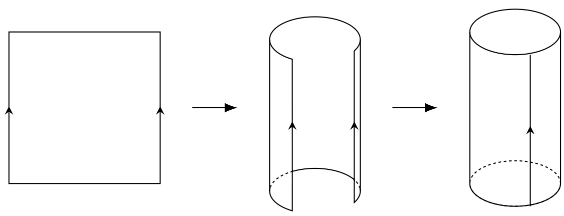


More generally, let $A \subseteq X$ be a subset of a topological space. This subset naturally induces the equivalence relation defined by $x \sim y$ if and only if $x = y$ or $\{x,y\} \subseteq A$, so every point of A is identified into a single equivalence class, while the points $x \in X \setminus A$ have singleton equivalence classes $[x] = \{x\}$. By an abuse of notation, we write X/A for the corresponding quotient space where all the points of A are identified into one point.

Example. Consider the square I^2. Define an equivalence relation by $(x,y) \sim (x',y')$ if and only if $(x,y) = (x',y')$ or $y = y'$ and $\{x,x'\} = \{0,1\}$. That is, we identify points on the left boundary with points on the right boundary with the same y-value. Visually, we represent this by marking an arrow on the square:



Then, we may identify marked edges together, with the arrows pointing in the same direction:



And we can see that the quotient space $I^2/\sim$ is homeomorphic to a cylinder (without the end faces).

The quotient map is an example of an *identification map* – a continuous surjective function $f : X \to Y$ between topological spaces X and Y such that $U \subseteq Y$ is open if and only if $f^{-1}[U] \subseteq X$ is open.

The reverse direction follows from continuity, so a identification map may also be characterised as a surjective map which also preserves open sets under direct images. Or put another way, $f : X \to Y$ is an identification map if and only if Y has the finest topology such that f is continuous (the *final topology* with respect to f).

Theorem 2.2. *A surjective map $f : X \to Y$ is an identification map if and only if for every space Z and every function $g : Y \to Z$, $g \circ f$ is continuous if and only if g is continuous.*

3 Compactness

A *cover* of a set A is a collection $\mathcal{U}$ of sets whose union contains A. That is,

$$A \subseteq \bigcup_{U \in \mathcal{U}} U$$

and we say that the elements of $\mathcal{U}$ *cover* A. A *subcover* of a cover $\mathcal{U}$ is a subset of $\mathcal{U}$ whose elements still cover A. A cover is *open* if every element of the cover is open.

Example.

- $\mathcal{U} = \{(n-2,n+2) : n \in \mathbb{Z}\}$ is an (open) cover of $\mathbb{R}$, with one possible subcover given by $S = \{(n-2,n+2) : n \in 2\mathbb{Z}\}$;
- $\mathcal{U} = \{(n,n+1) : n \in \mathbb{Z}\}$ is not a cover of $\mathbb{R}$ since it does not cover the integers.

A topological space T is *compact* if every open cover of T has a finite subcover.

Example.

- $(0,1)$ is not compact because $\mathcal{U} = \{(0,a) : a \in (0,1)\}$ is an open cover with no finite subcover;
- $\mathbb{R}$ is not compact because $\mathcal{U} = \{(-\infty,a), a \in \mathbb{R}\}$ has no finite subcover.

Note that, because compactness depends only on the open sets of a topological space, it is a topological invariant.

3.1 Lebesgue Numbers

Let $\mathcal{U}$ be an open cover of a metric space (X,d). A number $\delta > 0$ is called a *Lebesgue number* for $\mathcal{U}$ if for every $x \in X$, there exists an open set $U \in \mathcal{U}$ such that $\mathbb{B}(x,\delta) \subseteq U$.

In general, open covers do not have a Lebesgue number. For instance, $\mathcal{U} = \left\{(\frac{x}{2},x) : x \in (0,1)\right\}$ form an open cover of $(0,1)$, but the covering sets become arbitrary small as $x \to 0$, so no Lebesgue number exists.

Lemma (Lebesgue's Number Lemma)**.** *Every open cover $\mathcal{U}$ of a compact metric space (X,d) has a Lebesgue number.*

4 Diagrams

The structure of a collection of objects and morphisms (sets and set functions, topological spaces and continuous maps, etc.) is often visually represented as a directed graph, called a *diagram*. We are already familiar with the notation $A \to B$ to denote a morphism from A to B, but we can also draw larger diagrams with more objects and morphisms to represent more structure at once. For instance, this diagram depicts 3 objects with morphisms between them:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

A diagram is *commutative* if for every pair of objects in the diagram, all routes between them are equal. For instance, the diagram above is commutative if and only if $h = g \circ f$. This also justifies the omission of identity morphisms in general diagrams; they don't meaningfully add any additional paths to the diagram.

4.1 Isomorphisms

Suppose we have objects A and B and morphisms $f : A \to B$ and $g : B \to A$ such that the following diagram is commutative:

$$\begin{tikzcd} A \arrow["\mathrm{id}_A"', loop left] \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g"] \arrow["\mathrm{id}_B"', loop right] \end{tikzcd}$$

That is, $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$, so f and g are *mutually inverse*. Then, we say that f and g are *isomorphisms*, and we alternatively label g by f^{-1}. If an isomorphism between a pair of objects A and B exists, we say that A and B are *isomorphic* and we write $A \cong B$.

Isomorphic objects are, as far as the ambient category is concerned, effectively identical – anything you can say about one object will apply just as well to any other isomorphic object.

5 The Fundamental Problem

The *fundamental problem* in topology is to classify topological spaces up to homeomorphism. That is, given two topological spaces X and Y, can we determine whether $X \cong Y$ or not?

To show that two spaces are homeomorphic, one only needs to provide a homeomorphism. To prove that they are *not* homeomorphicm is much more difficult. This involves finding a property that is invariant under homeomorphism that is satisfied by one space, but not the other.

Example. $\{*\} \ncong \mathbb{R}$ because $\{*\}$ is finite (bounded, countable, compact, etc.), but $\mathbb{R}$ is not.

Example. $\mathbb{R} \ncong \mathbb{R}^2$ because $\mathbb{R}$ can be disconnected by removing one point, but $\mathbb{R}^2$ cannot.

But what about $\mathbb{R}^2$ and $\mathbb{R}^3$? Or $\mathbb{R}^3$ and $\mathbb{R}^4$? Or more generally, $\mathbb{R}^n$ and $\mathbb{R}^m$?

Compactness and cut-point arguments don't work in the general case, and most other topological invariants we have seen are also not sufficiently powerful to distinguish these spaces. One might think these pairs of spaces are not homeomorphic, as one feels "bigger"; but set-theoretically, they all have the same cardinality (apart from $\mathbb{R}^0 \cong \{*\}$). It turns out that showing that two real vector spaces are not homeomorphic is non-trivial.

Theorem 5.1 (Invariance of Domain, Brouwer 1912)**.** $\mathbb{R}^n \cong \mathbb{R}^m$ *if and only if* $m = n$*.*

We will develop some tools that will allow us to prove a partial version of this theorem in low dimensions.

5.1 Retractions

A *pair* (X,A) consists of a topological space X and a subspace $A \subseteq X$. When $A = \{x\}$ is a single point, we instead write (X,x), and call the pair a *pointed space* (we sometimes call X alone a pointed space with *basepoint* x).

A *map of pairs* $f : (X,A) \to (Y,B)$ is a continuous function $f : X \to Y$ such that $f(A) \subseteq B$. If A and B are points, then f is a *pointed* or *based* map.

A subset $A \subseteq X$ is a *retract* of X if there is a map $r : X \to A$, called the *retraction*, such that

$$r\big|_A = \mathrm{id}_A$$

That is, r surjects X onto A while keeping all points of A fixed.

Example. For any pointed space (X,x_0), the unique constant map $r : X \to \{x_0\}$ is a retraction.

Example. $\mathbb{R}^2 \setminus \{0\}$ retracts to S^1 via $r(x) = \frac{x}{\|x\|}$.

Example. I does not retract to $\{0,1\}$, as the continuous image of a connected space must be connected.

The following generalisation is non-trivial, and we will only be able to prove the $n = 2$ case later.

Theorem 5.2 (Brouwer)**.** *The disc D^n does not retract to S^{n-1}*

A subset $A \subseteq X$ is a (*strong*) *deformation retract* of X if there exists a one-parameter family of maps $f_t : X \to X$, $t \in I$ (or by uncurrying, a single map $F : X \times I \to X$), such that

- $f_0 = \mathrm{id}_X$;
- $f_1(X) = A$;
- $f_t\big|_A = \mathrm{id}_A$ for all $t \in I$.

Or, for all $x \in X$ and $a \in A$,

- $F(x,0) = x$;
- $F(X,1) = A$;
- $F(a,t) = a$ for all $t \in I$;

(A *weak deformation retract* relaxes the final condition for only $t = 1$. We will take the unqualified term "deformation retract" to always refer to the strong case.) Note that, by construction, f_1 is a retraction from X to A.

Example. $\mathbb{R}^n$ retracts to 0 via $F(x,t) = (1-t)x$. This is the *straight-line* or *linear homotopy.*

Example. $\mathbb{R}^n \setminus \{0\}$ deformation retracts to S^{n-1} via $F(x,t) = (1-t)x + t\frac{x}{\|x\|}$

Intuitively, a deformation retract continuously shrinks a space onto a subspace; as the parameter t increases, the image of F continuously transitions from all of X to only all of A, with A being fixed throughout the entire process.

We can also view F as a kind of mapping between the retraction f_1 and the identity $f_0 = \mathrm{id}_X$ on X, smoothly transforming one map to the other – and in fact, this kind of parametrised deformation between two maps defines a construction called a *homotopy.*

5.2 Homotopy

Let X and Y be topological spaces. A (*free*) *homotopy* is a continuous map $F : X \times I \to Y$. If $f_t(x) = F(x,t)$, then we say that F is a homotopy from f_0 to f_1. Two maps $f,g : X \to Y$ are *homotopic* if there exists a homotopy $F : X \times I \to Y$ such that $f = f_0$ and $g = f_1$, and we write $f \simeq g$ to denote this relation.

Theorem 5.3. *Homotopy is an equivalence relation on the set of continuous maps between two given topological spaces. That is, if $f,g,h : X \to Y$ are continuous maps, then*

(*i*) $f \simeq f$;

(*ii*) *If $f \simeq g$, then $g \simeq f$;*

(*iii*) *If $f \simeq g$ and $g \simeq h$, then $f \simeq h$.*

Proof.

(*i*) The constant homotopy $F(x,t) = f(x)$ is a homotopy between f and f.

(*ii*) If F is a homotopy from f to g, then $F\big(-,(1-t)\big)$ is a homotopy from g to f.

(*iii*) If F is a homotopy from f to g and G is a homotopy from g to h, then

$$H(x,t) = \begin{cases} F(x,2t) & t \leq \frac{1}{2} \\ G(x,2t-1) & t > \frac{1}{2} \end{cases}$$

is a homotopy from f to h, with continuity given by the pasting lemma.

∎

Recall that two spaces X and Y are homeomorphic if there exist a pair of maps between them with compositions equal to identities:

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y$$

$$g \circ f = \mathrm{id}_X \qquad \text{and} \qquad f \circ g = \mathrm{id}_Y$$

If we relax these conditions and only require that these compositions are *homotopic* to identities, then we obtain a weaker notion of likeness called *homotopy equivalence*:

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y$$

$$g \circ f \simeq \mathrm{id}_X \qquad \text{and} \qquad f \circ g \simeq \mathrm{id}_Y$$

We also say that f and g are *homotopy inverse* to one another.

Equality induces homotopy, but not the converse, so homeomorphic spaces are homotopy equivalent, but not the converse. More importantly, this means that two spaces that are not homotopy equivalent cannot be homeomorphic, allowing us another method to prove that two spaces are topologically distinguishable.

A space is always homotopy equivalent to any of its deformation retracts.

Example. $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$, as S^{n-1} is a deformation retract of $\mathbb{R}^n$.

In more detail, the homotopy equivalence is witnessed by the inclusion mapping $f : S^{n-1} \hookrightarrow \mathbb{R}^n \setminus \{0\}$ and the retract $g : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ defined by $x \mapsto \frac{x}{\|x\|}$. Then, $g \circ f = \mathrm{id}_{S^{n-1}}$, and $f \circ g$ is homotopic to $\mathrm{id}_{\mathbb{R}^n}$ via the straight-line deformation retract $F(x,t) = (1-t)x + t\frac{x}{\|x\|}$ found earlier.

Theorem 5.4. *Homotopy equivalence is an equivalence relation on the class of topological spaces.*

Proof. Symmetry and reflexivity are obvious. For transitivity, suppose $X \simeq Y$ and $Y \simeq Z$ witnessed by maps

$$X \underset{g_1}{\overset{f_1}{\rightleftarrows}} Y \underset{g_2}{\overset{f_2}{\rightleftarrows}} Z$$

and homotopies F_1 from $f_1 \circ g_1$ to id_Y and F_2 from $f_2 \circ g_2$ to id_Z. (Note that this diagram does not necessarily commute.)

Then, $f = f_2 \circ f_1$ and $g = g_1 \circ g_2$ are homotopy equivalence maps, with the homotopy from $f \circ g$ to id_Z given by

$$F(z,t) = \begin{cases} f_2 \circ F_1(z,2t) \circ g_2 & t \leq \frac{1}{2} \\ F_2(z,2t-1) & t > \frac{1}{2} \end{cases}$$

∎

A topological space X is *contractible* if $X \simeq \{*\}$. Or equivalently, if id_X is homotopic to a constant map (is *null-homotopic*).

Example. Euclidean space of any dimension is contractible: $\mathbb{R}^n \simeq \mathbb{R}^0$ for any $n \in \mathbb{N}$.

Consider the unique constant map $f : \mathbb{R}^n \to \mathbb{R}^0$ defined by $x \mapsto 0$ and the inclusion map $g : \mathbb{R}^0 \hookrightarrow \mathbb{R}^n$.

Then, we have $f \circ g = \mathrm{id}_{\mathbb{R}^0}$, and the composition $g \circ f : \mathbb{R}^n \to \mathbb{R}^n$ maps everything to zero, and is homotopic to the identity via the straight-line homotopy

$$F : \mathbb{R}^n \times I \to \mathbb{R}^n : (x,t) \mapsto tx$$

Example. Because $\{*\} \cong \mathbb{R}^0$, the above implies that $\mathbb{R}^n \simeq \mathbb{R}^m$ for all n,m. More generally, for any topological space X, $X \times \mathbb{R}^n \simeq X$.

In general, it is much more difficult to show that a space is not contractible. For instance, the proof that S^n is not contractible for any $n \geq 1$ is non-trivial.

We can also compare the notions of contractibility with that of deformation retracts.

Theorem 5.5. *If X deformation retracts to a point $x_0 \in X$, then it is contractible.*

Proof. Consider the retraction given by the unique constant map $f : X \to \{x_0\}$, and the inclusion mapping $g : \{x_0\} \hookrightarrow X$. We have $f \circ g = \mathrm{id}_{\{x_0\}}$, and $g \circ f = f_1$ and $\mathrm{id}_X = f_0$, where f_t is the deformation retraction. The deformation retraction then gives the required homotopy. ∎

Note that the converse does not hold, as the ordinary free homotopy demanded by a contractible space does not have to keep x_0 fixed throughout the homotopy.

Theorem 5.6. *The sphere S^n is not contractible for any $n \geq 0$.*

This theorem is highly non-trivial; we will only be able to prove the case $n = 1$ using the homotopy theory developed here.

5.3 Paths

Let X be a topological space, and $x,y \in X$ be two points. A *path* from x to y is a continuous map $f : I \to X$ with $f(0) = x$ and $f(1) = y$. We can view $f(s)$ as the position of a particle traveling along some curve in X as s varies from 0 to 1.

Note however, that a path is distinct from its image, and in particular, may not be injective. For instance, the image of the path $s \mapsto \exp(4\pi i s)$ in $S^1 \subset \mathbb{C}$ is the circle S^1, but the path itself travels around the circle twice and is distinct from, for example, the path $s \mapsto \exp(2\pi i s)$.

Given two paths $f,g : I \to X$ with $f(1) = g(0)$, the path $f * g : I \to X$ defined by

$$(f * g)(s) = \begin{cases} f(2s) & s \in [0,\frac{1}{2}] \\ g(2s-1) & s \in [\frac{1}{2},1] \end{cases}$$

is called the *concatenation* of f and g, with continuity given by the pasting lemma. Intuitively, the concatenation traverses the first path at double speed, then the second path at double speed.

Given a path $f : I \to X$ from x to y, the *reverse path* $\overline{f}$ defined by $\overline{f}(s) = f(1-s)$ is the path from y to x obtained by traversing f in the opposite direction.

We will often change the arguments to a map in concatenations and other similar operations, so it is helpful to be able to rescale any interval $[a,b]$ to $[0,1]$. This can be done via the affine map

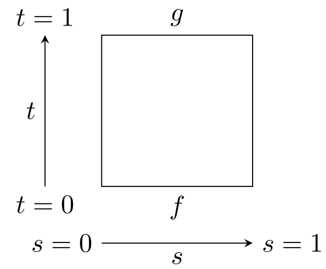
$$f(x) = \frac{x-a}{b-a}$$

i.e. translate down by a to reach zero, then rescale by the difference to reach 1.

Example. In the above concatenation, we have the intervals $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$. The rescaled argument of f is given by $\left(\frac{1}{2}-0\right)^{-1}(s-0) = 2s$, and of g by $\left(1-\frac{1}{2}\right)^{-1}\left(s-\frac{1}{2}\right) = 2s-1$.

Because paths are maps between topological spaces, we can also consider homotopies of paths. Given two paths $f,g : I \to X$, a homotopy between them is given by a map $F : I \times I \to X$ satisfying $f_0 = f$ and $f_1 = g$.

Because the domain of such a homotopy is a square $I \times I$, it can be visualised as



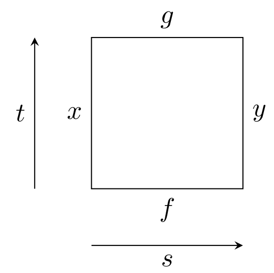
Each horizontal slice of the square at represents one of the functions f_t, with the bottom edge being f and the top edge being g, while each vertical slice represents the trajectory of a fixed argument s under the continuous deformation from f to g. This representation isn't very interesting yet, but will become helpful once we consider homotopies of concatenations.

The notion of free homotopy is, however, too weak to be very useful, since every path is homotopic to a constant path (i.e. deformation retract to the constant map at any point on the path), and we don't get much useful information from this. Instead, we can consider only paths that share endpoints, and define a more restricted notion of homotopy.

Let $x,y \in X$ and $f,g : I \to X$ be paths from x to y. In contrast to a free homotopy, a homotopy *relative to the boundary* or *endpoints* (sometimes abbreviated to "*rel boundary*") from f to g is a homotopy $F : I \times I \to X$ satisfying

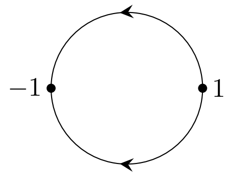
- $f_0 = f$;
- $f_1 = g$;
- $f_t(0) = x$ for all $t \in I$;
- $f_t(1) = y$ for all $t \in I$.

or as a diagram,



That is, a homotopy relative to boundaries is a continuous deformation of one path to another that keeps the endpoints of the paths fixed for all values of the parameter t. If there exists a homotopy relative to boundaries between f and g, then we write $f \overset{\partial}{\simeq} g$ to denote this relation.

Example. The paths $f,g : I \to S^1$ defined by $f(s) = \exp(\pi i s)$ and $g(s) = \exp(-\pi i s)$ traverse the upper and lower halves of the circle, respectively.



These paths are homotopic, as both can deformation retract to, for example, the point 1. However, they are not homotopic relative to boundaries. Intuitively, there is no way to continuously deform one to the other due to the hole in the circle that the two paths enclose. Proving this formally, however, is difficult.

Lemma 5.7. *For any pair of points $x,y \in X$, relative homotopy is an equivalence relation on the set of paths from x to y.*

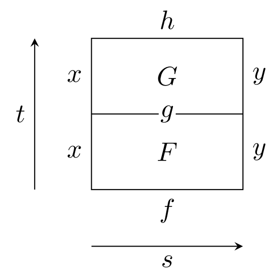
Proof. The proof is almost identical to that of free homotopy:

(i) The constant homotopy is a homotopy relative to boundaries from a path to itself.

(i) If F is a relative homotopy from f to g, then $F\big(-,(1-t)\big)$ is a relative homotopy from g to f.

(i) If F is a relative homotopy from f to g and G is a relative homotopy from g to h, then

$$H(s,t) = \begin{cases} F(s,2t) & t \leq \frac{1}{2} \\ G(s,2t-1) & t > \frac{1}{2} \end{cases}$$



is a relative homotopy from f to h.

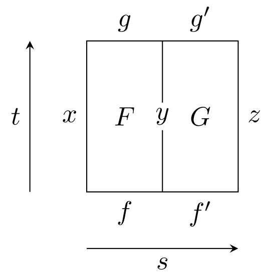
■

Lemma 5.8. *Let $f,g : I \to X$ be paths from x to y satisfying $f \overset{\partial}{\simeq} g$ and $f',g' : I \to X$ be paths from y to z satisfying $f \overset{\partial}{\simeq} g$. Then, $f * f' \simeq g * g'$.*

Proof. The proof is identical to that of transitivity in the previous lemma with the roles of s and t reversed.

If F is a relative homotopy from f to g and G is a relative homotopy from g to h, then

$$H(s,t) = \begin{cases} F(2s,t) & t \leq \frac{1}{2} \\ G(2s-1,t) & t > \frac{1}{2} \end{cases}$$



is a relative homotopy from $f * f'$ to $g * g'$. ■

More generally, a homotopy between maps $f,g : Z \to X$ may be relative to any subspace $A \subseteq X$. That is, the homotopy fixes the elements of the subspace A, and we write $f \overset{A}{\simeq} g$ if such a homotopy exists. A homotopy relative to boundaries is then the special case where the subspace consists of the two endpoints of the paths involved.

If we write $\iota : A \hookrightarrow X$ for the inclusion of A into X, then a deformation retract is just a special case of a retraction $r : X \to A$ such that $\iota \circ r$ is homotopic to id_X, relative to A.

5.4 Loops

A *loop* is a special case of a path where the two endpoints coincide. That is, a continuous map $f : I \to X$ with $f(0) = f(1) = x_0 \in X$, and we say that f is a loop *based at* x_0 or *with basepoint* x_0.

Because loops are a special case of paths, homotopy relative to boundaries is also an equivalence relation on the set of loops at some basepoint, so given a fixed point x_0, we can form equivalence classes of the form

$$[f] = \left\{(g : I \to X) : g(0) = g(1) = x_0, g \overset{\partial}{\simeq} f\right\}$$

A homotopy relative to boundaries between loops is also called a *based homotopy*, since the preserved subspace is a single point, as in a pointed space.

Given a pointed space (X,x_0), we denote the set of homotopy classes of loops based at x_0 as

$$\pi_1(X,x_0) = \left\{[f] : f(0) = f(1) = x_0\right\}$$

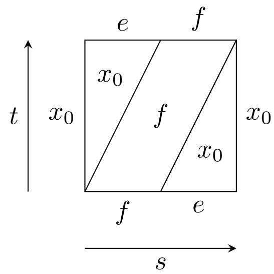
The concatenation of two loops based at x_0 is also a loop based at x_0, and we also have that if $f \simeq g$ and $f' \simeq g'$, then $f * f' \simeq g * g'$, so concatenation is compatible with homotopy. This allows us to define an operation

$$[f] \bullet [g] \coloneqq [f * g]$$

For any pointed space (X,x_0), the set $\pi_1(X,x_0)$ equipped with this operation forms a group, called the *fundamental group* or *first homotopy group* of (X,x_0).

Theorem 5.9. *For any pointed space* (X,x_0)*, the* $\big(\pi_1(X,x_0),\bullet\big)$ *is a group, with unit* $[e]$*, where* e *is the constant loop, and the inverse* $[f]^{-1}$ *of the element* $[f]$ *is the class* $[\overline{f}]$*, where* $\overline{f}(t) = f(1-t)$ *is the inverse loop.*

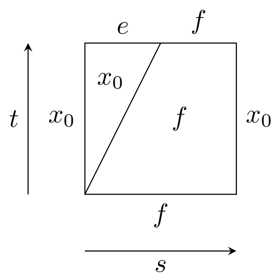
Proof. A homotopy showing $f * e \overset{\partial}{\simeq} e * f$ can be given as a diagram



To find the equation for this homotopy, we find the interval where f is applied to s; $[\frac{t}{2},\frac{1+t}{2}]$, as the homotopy will be constant outside of this interval; then find the affine function that varies from 0 to 1 as s varies from $\frac{t}{2}$ to $\frac{1+t}{2}$; $\left(\frac{1+t}{2} - \frac{t}{2}\right)^{-1}\left(s - \frac{t}{2}\right) = 2s - t$:

$$F(s,t) = \begin{cases} x_0 & s \in [0,\frac{t}{2}] \\ f(2s-t) & s \in [\frac{t}{2},\frac{1+t}{2}] \\ x_0 & s \in [\frac{1+t}{2},1] \end{cases}$$

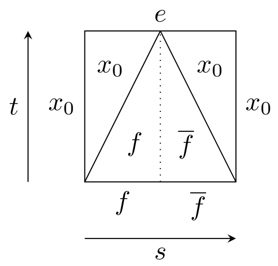
The homotopy $e * f \overset{\partial}{\simeq} f$ is then given by



We again find the argument, $\left(1 - \frac{t}{2}\right)^{-1}\left(s - \frac{t}{2}\right) = \frac{2s-t}{2-t}$, and then set the function to be constant past the linear bound:

$$F(s,t) = \begin{cases} x_0 & s \in [0,\frac{t}{2}] \\ f(\frac{2s-t}{2-t}) & s \in [\frac{t}{2},1] \end{cases}$$

For inverses, let $f : I \to X$ be a loop at x_0, and let $\overline{f} : I \to X$ be the loop defined by $\overline{f}(s) = f(1-s)$. Then, the homotopy $f * \overline{f} \overset{\partial}{\simeq} e$ is given by

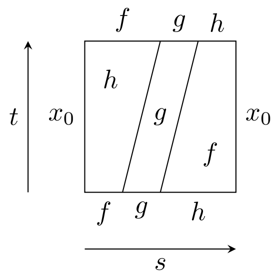


The concatenation $f * \overline{f}$ represents walking along f (at double speed), then walking back along the same path in reverse (also in double speed). Here, as t increases, the homotopy represents walking at the same speed, but along less and less of the path before returning (so no rescaling is performed this time):

$$F(s,t) = \begin{cases} x_0 & s \in [0,\frac{t}{2}] \\ f(2s-t) & s \in [\frac{t}{2},\frac{1}{2}] \\ f(2s-1+t) & s \in [\frac{1}{2},1-\frac{t}{2}] \\ x_0 & s \in [1-\frac{t}{2},1] \end{cases}$$

Since reversal is involutive, replacing f by $\overline{f}$ in the previous argument yields a homotopy $\overline{f} * f \overset{\partial}{\simeq} e$.

For associativity, a homotopy $(f * g) * h \overset{\partial}{\simeq} f * (g * h)$ is given by



For f, the argument is $\left(\frac{1+t}{4} - 0\right)^{-1}(s-0) = \frac{4s}{1+t}$; for g, $\left(\frac{2+t}{4} - \frac{1+t}{4}\right)^{-1}\left(s - \frac{1+t}{4}\right) = 4s-1-t$; and for h, $\left(1 - \frac{2+t}{4}\right)^{-1}\left(s - \frac{2+t}{4}\right) = \frac{4s-2-t}{2-t}$:

$$F(s,t) = \begin{cases} f(\frac{4s}{1+t}) & s \in [0,\frac{1+t}{4}] \\ g(4s-1-t) & s \in [\frac{1+t}{4},\frac{2+t}{4}] \\ h(\frac{4s-2-t}{2-t}) & s \in [\frac{2+t}{4},1] \end{cases}$$

■

5.5 The Fundamental Group

5.5.1 Path-Connected Spaces

A space X is *path-connected* if for every pair of points $x,y \in X$, there exists a path from x to y.

Theorem 5.10. *If X is path-connected, then for any two points $x_0,x_1 \in X$, we have an isomorphism of fundamental groups, $\pi_1(X,x_0) \cong \pi_1(X,x_1)$.*

Theorem 5.11. *For each path $h : I \to X$ from x_0 to x_1, define the map $\beta_h : \pi_1(X,x_0) \to \pi(X,x_1)$ by $[f] \mapsto [\overline{h} * f * h]$.*

Then,

$$\begin{aligned}\beta_h\big([f]\bullet[g]\big) &= \beta_h\big([f*g]\big)\\ &= [\overline{h}*f*g*h]\\ &= [\overline{h}*f*h*\overline{h}*g*h]\\ &= [\overline{h}*f*h]\bullet[\overline{h}*g*h]\\ &= \beta_h\big([f]\big)\bullet\beta_h\big([g]\big)\end{aligned}$$

so β_h is a group homomorphism for any path h. In particular, the map $\beta_{\overline{h}}$ induced by the reverse path is also a group homomorphism.

Because $h*\overline{h}\overset{\partial}{\simeq}e_{x_0}$ and $\overline{h}*h\overset{\partial}{\simeq}e_{x_1}$, we also have

$$\begin{aligned}\beta_{\overline{h}}\circ\beta_h\big([f]\big) &= [h*\overline{h}*f*h*\overline{h}]\\ &= [f]\\ &= \mathrm{id}_{\pi_1(X,x_0)}\big([f]\big)\end{aligned}$$

and similarly, $\beta_h\circ\beta_{\overline{h}}=\mathrm{id}_{\pi_1(X,x_1)}$, so β_h and $\beta_{\overline{h}}$ are inverse maps and hence form an isomorphism $\pi_1(X,x_0)\cong\pi_1(X,x_1)$.

Due to these isomorphisms, for path-connected spaces X, we may omit the basepoint and write just $\pi_1(X)$ for the fundamental group.

6 Covering Spaces

Let X be a topological space. A *covering* of X is a map $p:\tilde{X}\to X$ such that for every point $x\in X$, there exists an open neighbourhood $U_x\subseteq X$ of x and a discrete (i.e. every set is open) space $D_x\subseteq\tilde{X}$ such that

$$p^{-1}[U_x]=\bigsqcup_{d\in D_x}V_d$$

and the restriction $p\big|_{V_d}:V_d\to U_x$ is a homeomorphism for every $d\in D_x$. Such an open set U_x is said to be *evenly covered* by p, and the open sets V_d are called the *sheets* of the covering.

If $p:\tilde{X}\to X$ is a covering, then the pair $(\tilde{X},p)$ is called a *covering space* or *cover* of X, and X is said to be the *base* of the covering.

Intuitively, a covering is a surjective map that acts locally like a projection of multiple copies of a space onto itself.

The preimage $p^{-1}\big[\{x\}\big]$ of any point x is called the *fibre* of x. A covering $p:\tilde{X}\to X$ is called an *n-fold covering* if the fibre $p^{-1}\big[\{x\}\big]$ consists of n points for all $x\in X$.

Example. For any $k\in\mathbb{N}$, the map $p_k:S^1\to S^1$ defined by $z\mapsto z^k$ is a covering map. Given a point $z=\exp(2\pi it)\in S^1$, we take the open neighbourhood $U=\big\{\exp(2\pi is):|s-t|<\varepsilon\big\}$ for some $0<\varepsilon<\frac{1}{2k}$, which has preimage

$$\begin{aligned}p^{-1}[U] &= \left\{\sqrt[k]{\exp(2\pi is)},|s-t|<\varepsilon\right\}\\ &= \bigcup_{0\le j<k}\left\{\exp\!\left(2\pi i\frac{s+j}{k}\right):|s-t|<\varepsilon\right\}<\varepsilon\end{aligned}$$

These sets are all homeomorphic to $V = \{\exp(2\pi is/k) : |s-t| < \varepsilon\}$, and because $\frac{t+\varepsilon+j}{k} < \frac{t-\varepsilon+(j+1)}{k}$ for each j, they are all disjoint, so,

$$= \bigsqcup_{i=1}^{n} V$$

Intuitively, the preimage of the arc of length $2\varepsilon = \frac{1}{k}$ centred on z is the collection of arcs that each cover $\frac{1}{k}$th of the circle, centred on each root of z, and these arcs are disjoint as there are exactly k such roots evenly spaced along the circle.

This covering is also an k-fold covering map, as the fibre of any point $z = \exp(2\pi it)$ consists of k many kth roots of z – namely $\exp\bigl(2\pi i(t+j)/k\bigr)$, for $0 \le j < k$.

Example. The map $p_\infty : \mathbb{R} \to S^1$ defined by $x \mapsto \exp(2\pi ix)$ is a covering map. Given a point $z = \exp(2\pi it) \in S^1$, we take the open neighbourhood $U = \{\exp(2\pi is) : |s-t| < \varepsilon\}$ for some $0 < \varepsilon < 1$, which has preimage

$$\begin{aligned} p^{-1}[U] &= \bigcup_{j\in\mathbb{Z}} \{s+i : |s-t| < \varepsilon\} \\ &= \bigsqcup_{j\in\mathbb{Z}} V \end{aligned}$$

Two coverings $p : Y \to X$ and $q : Z \to X$ are *isomorphic* if they factor through each other. That is, there exist maps f and g such that

$$p = q \circ f \qquad \text{and} \qquad q = p \circ g$$

This also implies that f and g are inverse, so equivalently, p and q are isomorphic if there exists a homeomorphism $h : Y \to Z$ such that

$$\begin{tikzcd} Y \arrow[rr, "h", "\cong"'] \arrow[dr, "p"'] & & Z \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

commutes.

Example. p_2 is isomorphic to p_{-2} via the homeomorphism $h(z) = z^{-1}$.

Example. p_2 and p_3 are not isomorphic, as one is a 2-fold covering, and the other is a 3-fold covering.

Let $p : \tilde{X} \to X$ be a covering of X. A *deck transformation* is a homeomorphism $\tau : \tilde{X} \to \tilde{X}$ such that $p \circ \tau = p$. That is, τ witnesses an automorphism of p. The set of all deck transformations of a cover p is denoted $\mathrm{Deck}(p)$, and has group structure under composition.

Example. The map $z \mapsto -z$ is a deck transformation for p_2.

6.1 Liftings

Given a covering $p : \tilde{X} \to X$ and a map $f : Y \to X$, a *lift* of f is a map $\tilde{f} : Y \to \tilde{X}$ such that

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

commutes. That is, f factors through $\tilde{f}$.

Lemma 6.1. *Let $p : \tilde{X} \to X$ be a cover, and let $\tilde{f},\tilde{g} : Y \to \tilde{X}$ be continuous maps. Then,*

(i) *$\tilde{f}$ is a lift of $p \circ \tilde{f}$;*

(ii) *If $\tilde{f} \simeq \tilde{g}$, then $p \circ \tilde{f} \simeq p \circ \tilde{g}$ ("homotopies descend");*

(iii) *If $\alpha,\beta : I \to X$ are paths with $\alpha(1) = \beta(0)$, then $p \circ (\alpha * \beta) = (p \circ \alpha) * (p \circ \beta)$ ("paths descend").*

Proof.

(i) The diagram

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "p\circ\tilde{f}"'] & X \end{tikzcd}$$

trivially commutes.

(ii) Let $F : Y \times I \to \tilde{X}$ be a homotopy between $f_0 = \tilde{f}$ and $f_1 = \tilde{g}$. Then, $p \circ F : Y \times I \to X$ is a homotopy between $p \circ f_0 = p \circ \tilde{f}$ and $p \circ f_1 = p \circ \tilde{g}$.

(iii) Expanding the definition of concatenation, we have

$$\begin{aligned}\big(p \circ (\alpha * \beta)\big)(s) &= p \circ \begin{cases} \alpha(2s) & s \in [0,\frac{1}{2}] \\ \beta(2s-1) & s \in [\frac{1}{2},1] \end{cases} \\ &= \begin{cases} p \circ \alpha(2s) & s \in [0,\frac{1}{2}] \\ p \circ \beta(2s-1) & s \in [\frac{1}{2},1] \end{cases} \\ &= \big((p \circ \alpha) * (p \circ \beta)\big)(s) \end{aligned}$$

■

6.2 Homotopy Lifting Property

Let $p : Z \to X$ be a continuous map. Then, p has the *homotopy lifting property* (HLP) if for any homotopy $F : Y \times I \to X$ and lift $g : Y \times \{0\} \to Z$ of f_0 (i.e. $f_0 = p \circ g$), there exists a unique homotopy $\tilde{F} : Y \times I \to Z$ such that

(i) $\tilde{f}_0 = g$;

(ii) $p \circ \tilde{F} = F$.

That is,

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "g"] \arrow[d, hook, "\iota"] & Z \arrow[d, "p"] \\ Y \times I \arrow[r, "F"] \arrow[ur, dashed, "\tilde{F}"] & X \end{tikzcd}$$

commutes.

If we take $Y = \{*\}$ to be a singleton set, we may interpret the homotopies above as paths, and a lift $g : \{*\} \times \{0\} \to Z$ is simply a choice of a point in $p^{-1}\big[\{x_0\}\big]$:

Let $p : Z \to X$ be a continuous map. Then, p has the *path lifting property* (PLP) if for any path $f : I \to X$ with $f(0) = x_0$ and point $\tilde{x}_0 \in p^{-1}\big[\{x_0\}\big]$, there exists a unique path $\tilde{f} : I \to Z$ with

$\tilde{f}(0) = \tilde{x}_0$ and $p \circ \tilde{f} = f$.

$$\begin{tikzcd}
\{*\} \times \{0\} \cong \{*\} \arrow[r, "g"] \arrow[d, hook, "\iota"] & Z \arrow[d, "p"] \\
\{*\} \times I \cong I \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & X
\end{tikzcd}$$

6.2.1 The Local Homotopy Lifting Property

Let $\mathcal{U}$ be an open cover of a metric space (X,d).

The *diameter* of a subset $S \subseteq X$ is the least upper bound of the distance between any pair of points in that subset:

$$\operatorname{diam}(S) = \sup_{x,y \in S} d(x,y)$$

Recall that a number $\delta > 0$ is called a *Lebesgue number* for $\mathcal{U}$ if for every $x \in X$, there exists an open neighbourhood $U \in \mathcal{U}$ of x such that $\mathbb{B}(x,\delta) \subseteq U$.

Equivalently, $\delta > 0$ is a Lebesgue number for $\mathcal{U}$ if every subset $S \subseteq X$ with diameter at most $\operatorname{diam}(S) \leq \delta$ is contained within some member of the cover.

Lemma. *Let $\{I_\alpha\}_\alpha$ be an open cover of the unit interval I. Then, there exists a Lebesgue number for this cover. That is, there exists some $\delta > 0$ such that for every $S \subseteq I$ with diameter $\operatorname{diam}(S) \leq \delta$, we have $S \subseteq I_\alpha$ for some α.*

This is a special case of Lebesgue's number lemma (§ 3.1) applied to the unit interval.

Recall that, given a covering space $p : \tilde{X} \to X$, we can find a covering of X by evenly covered sets $\{U_\alpha\}_\alpha$ such that the preimage of each set U_α is a disjoint union of open sets $\{V_\alpha^\beta\}_\beta$

$$p^{-1}[U_\alpha] = \bigsqcup_\beta V_\alpha^\beta$$

and furthermore, the restrictions of the covering to each of these sets is a homeomorphism

$$p\big|_{V_\alpha^\beta} : V_\alpha^\beta \overset{\cong}{\to} U_\alpha$$

wth inverses denoted by $q_\alpha^\beta : U_\alpha \to V_\alpha^\beta$.

Let $F : Y \times I \to X$ be a homotopy and $g : Y \times \{0\} \to \tilde{X}$ be a lift of f_0, and suppose that the image of F is contained within an evenly covered subset $U_\alpha \subseteq X$. If the lift carries the domain of f_0 to one of the sheets V^β – that is, if $g(Y \times \{0\}) \subseteq V^\beta$ – then we can lift the whole homotopy F to a homotopy $\tilde{F} := q_\alpha^\beta \circ F$ that extends g.

Lemma 6.2. *Let $p : \tilde{X} \to X$ be a covering, and let $F : Y \times I \to X$ be a homotopy. Let $g : Y \times \{0\} \to \tilde{X}$ satisfy $p \circ g = f_0$. Then, for every $y_0 \in Y$, there exists an open neighbourhood $N \subseteq Y$ and a unique homotopy $\tilde{F}_N : N \times I \to \tilde{X}$ such that*

- $p \circ \tilde{F}_N = F\big|_{N \times I}$;
- $\tilde{F}_N(-,0) = g\big|_{N \times \{0\}}$

Moreover, if $M \subseteq Y$ is another such neighbourhood of y_0, then

$$\tilde{F}_M\big|_{(M \cap N) \times I} = \tilde{F}_N\big|_{(M \cap N) \times I} = \tilde{F}_{M \cap N}$$

Theorem 6.3. *Covering maps satisfy the homotopy lifting property*

Proof. Let $p : \tilde{X} \to X$ be a covering of X and $f : Y \to X$ be continuous. Cover $Y \times I$ with open sets $N_\alpha \times I$ as in the previous lemma.

This yields a family of lifts $\tilde{F}_{N_\alpha} : N_\alpha \times I \to \tilde{X}$ that coincide on the intersection of any two sets $N_i \times I$ and $N_j \times I$ in the cover, and hence we have a well-defined function $\tilde{F} : Y \times I \to \tilde{X}$ defined by piecing these lifts together. Since each local lift is continuous, $\tilde{F}$ is continuous by the pasting lemma, and is therefore itself a lift.

Uniqueness follows from the uniqueness of the homotopy given by the previous lemma. ■

6.3 The Fundamental Group of the Circle

Lemma 6.4. *The map $\Phi : \mathbb{Z} \to \pi_1(S^1{,}1)$ defined by $n \mapsto [\omega_n]$ is a group homomorphism.*

Proof. The map $\tilde{\omega}_n : I \to \mathbb{R}$ defined by $t \mapsto nt$ satisfies

$$\omega_n = p_\infty \circ \tilde{\omega}_n$$

so it is a lift of ω_n. Define the deck transformation $\tau : \mathbb{R} \to \mathbb{R}$ by $t \mapsto t + n$ and consider the composition $\tilde{\omega}_m \cdot (\tau_m \circ \tilde{\omega}_n)$. This composition is a path in $\mathbb{R}$ from 0 to $m + n$, and is therefore homotopic to $\tilde{\omega}_{m+n}$ (e.g. via the straight-line homotopy).

Then,

$$\begin{aligned}
\Phi(m+n) &= [\omega_{m+n}] \\
&= [p_\infty \circ \tilde{\omega}_{m+n}] \\
&= \big[p_\infty \circ \big(\tilde{\omega} \cdot (\tau \circ \tilde{\omega}_n)\big)\big] \\
&= [p_\infty \circ \tilde{\omega}_m \cdot p_\infty \circ \tau_m \circ \tilde{\omega}_n] \\
&= [p_\infty \circ \tilde{\omega}_m] \bullet [p_\infty \circ \tau_m \circ \tilde{\omega}_n] \\
&= [p_\infty \circ \tilde{\omega}_m] \bullet [p_\infty \circ \tilde{\omega}_n] \\
&= [\omega_m] \bullet [\omega_n] \\
&= \Phi(m) \bullet \Phi(n)
\end{aligned}$$

■

Theorem 6.5. *The map $\Phi : \mathbb{Z} \to \pi_1(S^1{,}1)$ defined by $n \mapsto [\omega_n]$ is a group isomorphism.*

Proof. By the path lifting property of covers, given a loop $\alpha \in S^1$, there exists a unique lift $\tilde{\alpha} : I \to \mathbb{R}$ such that

(*i*) $p \circ \tilde{\alpha} = \alpha$;

(*ii*) $\tilde{\alpha}(0) = 0$.

Since $\alpha(1) = 1$ and $p \circ \tilde{\alpha} = \alpha$, we have $\tilde{a}(1) \in p^{-1}\big[\{1\}\big] = \mathbb{Z}$. Denote this value by $n := \tilde{\alpha}(1)$. We then have $\tilde{\alpha} \overset{\partial}{\simeq} \tilde{\omega}_n$ since both are paths from 0 to n in $\mathbb{R}$, with a homotopy given by the straight-line homotopy. Since homotopies descend,

$$\alpha = p_\infty \circ \tilde{\alpha} \overset{\partial}{\simeq} p_\infty \circ \omega_n = \omega$$

so $[\alpha] = [\omega_n]$ and Φ is surjective.

Now, suppose that $\Phi(n) = [\omega_n] = [e]$. That is, $\omega_n \overset{\partial}{\simeq} e$, given by a homotopy $F : I \times I \to S^1$ from $f_0 = \omega_n$ to $f_1 = e$.

Define a map $g : I \times \{0\} \to \mathbb{R}$ by $g(s,0) = \tilde{\omega}_n$. The cover p then lifts F to a homotopy $\tilde{F} : I \times I \to R$ from $\tilde{f}_0 = g$ to $p \circ \tilde{F} = F$. The other end of the homotopy $\tilde{f}_1$ then satisfies $p \circ \tilde{f}_1 = e$, the constant loop. Thus,

- $\tilde{f}_0(0) = 0$ since $\tilde{f}_0 = \tilde{\omega}_n$;
- $\tilde{f}_0(1) = n$ since $\tilde{f}_0 = \tilde{\omega}_n$;
- $\tilde{f}_t(0) \in \mathbb{Z}$ since $p_\infty \circ \tilde{f}_t(0) = f_t(0) = 1$;
- $\tilde{f}_t(1) \in \mathbb{Z}$ since $p_\infty \circ \tilde{f}_t(1) = f_t(1) = 1$;
- $\tilde{f}_1(s) \in \mathbb{Z}$ since $p_\infty \circ \tilde{f}_1(s) = e(s) = 1$;

As the continuous image of a connected space is connected, any continuous map that takes values in $\mathbb{Z} \subseteq \mathbb{R}$ must be constant. Thus,

$$\begin{aligned} 0 &= \tilde{f}_0(0) \\ &= \tilde{f}_t(0) \\ &= \tilde{f}_1(s) \\ &= \tilde{f}_t(1) \\ &= \tilde{f}_0(1) \\ &= n \end{aligned}$$

so Φ has trivial kernel. ■

7 Induced Homomorphisms

Recall that a pair of spaces (X,A) consists of two topological spaces satisfying $A \subseteq X$, and that a map of pairs $f : (X,A) \to (Y,B)$ is a continuous function $f : X \to Y$ such that $f(A) \subseteq B$, also called a pointed or based map when A and B are singletons.

The *induced homomorphism* of a pointed map $f : (X,x_0) \to (Y,y_0)$ is the map

$$\begin{aligned} f_* : \pi_1(X,x_0) &\to \pi_1(Y,y_0) \\ [\alpha] &\mapsto [f \circ \alpha] \end{aligned}$$

Lemma 7.1. *The induced homomorphism f_* is a group homomorphism.*

Proof. Let $\alpha \overset{\partial}{\simeq} \beta$, witnessed by $F : I \times I \to X$. Then, $G = f \circ F$ is a relative homotopy from $f \circ \alpha$ to $f \circ \beta$, so we have $f \circ \alpha \overset{\partial}{\simeq} f \circ \beta$, and the map is well-defined.

Now, let α,β be loops in $\pi(X,x_0)$. Then,

$$\begin{aligned} f_*\big([\alpha] \bullet [\beta]\big) &= f_*\big([\alpha * \beta]\big) \\ &= \big[f \circ (\alpha * \beta)\big] \\ &= \big[(f \circ \alpha) * (f \circ \beta)\big] \\ &= [f \circ \alpha] * [f \circ \beta] \\ &= f_*\big([\alpha]\big) \bullet f_*\big([\beta]\big) \end{aligned}$$

■

Example. Consider the covering map $p_2 : (S^1,1) \to (S^1,1)$ defined by $z \mapsto z^2$, and let ω_n be the loop defined by $\omega_n(s) = \exp(2\pi i n s)$. Then, $p_2 \circ \omega_n = \omega_{2n}$, so the induced homomorphism $(p_2)_*$ is defined by

$$(p_2)_*\big([\omega_n]\big) = [\omega_{2n}]$$

Theorem 7.2. *Induced homomorphisms satisfy the following properties:*

(*i*) $\big(\mathrm{id}_{(X,x_0)}\big)_* = \mathrm{id}_{\pi_1(X,x_0)}$;

(*ii*) *Given two pointed maps* $f : (X,x_0) \to (Y,y_0)$ *and* $g : (X,x_0) \to (Y,y_0)$, *we have,*

$$(g \circ f)_* = g_* \circ f_*$$

That is, the fundamental group is a functor $\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$, *acting on objects by* $(X,x_0) \mapsto \pi_1(X,x_0)$ *and on morphisms by* $f \mapsto f_*$.

Proof.

(*i*) Precomposing by the identity leaves the loop unchanged, and thus the fundamental group is unchanged.

(*ii*) Given a loop γ, we have

$$\begin{aligned}(g \circ f)_*\big([\gamma]\big) &= \big[(g \circ f) \circ \gamma\big] \\ &= \big[g \circ (f \circ \gamma)\big] \\ &= g_*\big([f \circ \gamma]\big) \\ &= (g_* \circ f_*)\big([\gamma]\big)\end{aligned}$$

■

Theorem 7.3. *If* $f : (X,x_0) \to (Y,y_0)$ *is an isomorphism, then* $f_* : \pi_1(X,x_0) \to \pi_1(Y,y_0)$ *is also an isomorphism.*

Proof. Follows from functoriality. That is,

$$\begin{aligned}\mathrm{id}_{\pi_1(X,x_0)} &= \big(\mathrm{id}_{(X,x_0)}\big)_* \\ &= \big(f \circ f^{-1}\big)_* \\ &= f_* \circ f_*^{-1}\end{aligned}$$

(and similarly with f and f^{-1} reversed). ■

It follows that the fundamental group of a path-connected space is a topological invariant: if $X \cong Y$, then $\pi_1(X) \cong \pi_1(Y)$.

8 Homotopy Invariance

Recall that, given a pair (X,A), a *retraction* is a map $r : X \to A$ such that $r\big|_A = \mathrm{id}_A$. Retractions and inclusions naturally fit together in a square,

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, hook, "\iota"'] & A \arrow[d, hook, "\iota"] \\ X \arrow[ur, "r"] \arrow[r, "\iota \circ r"'] & X \end{tikzcd}$$

noting that the upper triangle gives $r \circ \iota = \mathrm{id}_A$.

Theorem 8.1. *Let $r : X \to A$ be a retraction, $\iota : A \hookrightarrow X$ be the inclusion. Then, for any point $x_0 \in A$, the induced homomorphisms $r_* : \pi_1(X,x_0) \to \pi_1(A,x_0)$ and $\iota_* : \pi_1(A,x_0) \to \pi_1(X,x_0)$ have the following properties:*

(*i*) *r_* is surjective and ι_* is injective;*

(*ii*) *If r is a deformation retract, then r_* and ι_* constitute an isomorphism.*

Proof.

(*i*) Because $r \circ \iota = \mathrm{id}_{(A,x_0)}$, we have from functoriality of π_1 that $\mathrm{id}_{\pi_1(A,x_0)} = (r \circ \iota)_* = r_* \circ \iota_*$, so ι_* and r_* must be injective and surjective, respectively.

(*ii*) We have that r_* is surjective, so to establish an isomorphism, it suffices to show that r_* is also injective if it is a deformation retract.

Denote by $e_A : I \to A$ and $e_X : I \to X$ the constant loops at x_0 in A and X, respectively. Let $[\gamma] \in \pi_1(X,x_0)$, and suppose that $[\gamma] \in \ker(r_*)$, so $r_*\big([\gamma]\big) = [r \circ \gamma] = [e_A]$, or equivalently, $r \circ \gamma \overset{\partial}{\simeq} e_A$.

As $r \circ \gamma$ is a loop in $A \subseteq X$, postcomposing by the inclusion gives the loop $\iota \circ r \circ \gamma$ in X that is homotopic to e_X by the same homotopy that takes $r \circ \gamma$ to e_A.

Because r is a deformation retract, we have $\iota \circ r \overset{\partial}{\simeq} \mathrm{id}_X$ witnessed by a homotopy $F : X \times I \to X$ relative to A. Construct a new homotopy $G : I \times I \to X$ by $G(s,t) = F\big(\gamma(s),t\big)$ between $g_0 = \iota \circ r \circ \gamma$ and $g_1 = \gamma_1$. Note that G is a based homotopy (at the subspace $\{0 \sim 1\} \subset I$) since F is a homotopy relative to A, and $x_0 \in A$, so $g_t(0) = f_t(x_0) = x_0$ for all $t \in I$.

Then, we have $e_X \overset{\partial}{\simeq} \iota \circ r \circ \gamma \overset{\partial}{\simeq} \gamma$ so $[\gamma] = [e_X]$. It follows that r_* has trivial kernel and is thus injective.

Now, let $[\eta] \in \pi_1(X,x_0)$, and define a new based homotopy from F in the same way as before; $G(s,t) = F\big(\eta(s),t\big)$. Because $f_1(X) = r(X) = A$, the loop $g_1 = f_1 \circ \eta$ is contained within A, so g_1 is a loop in A and hence $[g_1] \in \iota_*\big(\pi_1(A,x_0)\big)$. Since G is a homotopy, $g_1 \overset{\partial}{\simeq} \eta$, so $[g_1] = [\eta]$, and ι_* is surjective.

■

In particular, this also implies that $(\iota \circ r)_* : \pi_1(X,x_0) \to \pi_1(X,x_0)$ is also an isomorphism for any deformation retract r.

Theorem 8.2. *If $f : X \to Y$ is a homotopy equivalence, then for any $x_0 \in X$, the induced homomorphism $f_* : \pi_1(X,x_0) \to \pi\big(Y,f(x_0)\big)$ is an isomorphism.*

This shows that not only is the fundamental group a *topological* invariant, but more generally a *homotopy* invariant: if $X \simeq Y$, then $\pi_1(X) \cong \pi_1(Y)$.

Proof. WIP ■

9 The Brouwer Fixed Point Theorem

In the previous section, we showed that retractions induce surjective homomorphisms. One simple application is as follows:

Theorem 9.1. *There is no retract from the the unit disc D^2 to the circle S^1.*

Proof. Such a retraction would imply a surjection $\pi_1(D^2,1) \twoheadrightarrow \pi_1(S^1,1)$, but $\pi_1(D^2,1) = 0$, while $\pi_1(S^1,1) \cong \mathbb{Z}$. ■

A more important consequence of this "no retract" theorem generalises the fact that a continuious function $f : I \to I$ has a fixed point. This is a straightforward consequence of the intermediate value theorem (in fact, the statement is true if f is only increasing, and not continuous, though this proof is more involved), but the generalisation to maps f from $I \times I \cong D^2$ to itself is surprisingly non-trivial.

Theorem (Brouwer Fixed Point Theorem)**.** *Every map $f : D^2 \to D^2$ has a fixed point.*

Proof. WIP ■

9.1 Applications

One application of the Brouwer fixed point theorem is to eigenvectors. The following result is a special case of the *Perron-Frobenius theorem*:

Theorem 9.2. *Let $\mathbf{A} \in \mathbb{R}^{3\times 3}$ be a matrix with only positive entries. Then, $\mathbf{A}$ has an eigenvector $\mathbf{v}$ with only positive entries.*

Proof. WIP ■

Another application of the Brouwer fixed point theorem is the famous *Borsuk-Ulam theorem*, but first, some additional theory is required for its proof.

9.1.1 Odd and Even Maps

Recall that an involution is an endofunction $f : X \to X$ that is its own inverse; $f\big(f(x)\big) = x$, or $f \circ f = \mathrm{id}_X$. One important example is the negation function, $f(x) = -x$ – but note that this only makes sense in spaces that are symmetric about the origin of their coordinate systems. For instance, $f(x) = -x$ does not make sense as function $I \to I$.

Let X and Y be spaces with negation. A map $f : X \to Y$ is *odd* if $f(-x) = -f(x)$, and *even* if $f(-x) = f(x)$ for all $x \in X$. Note that a map may be neither odd nor even.

Example.

- The zero map is the unique map that is simultaneously odd and even.
- The identity map is odd, as $\mathrm{id}(-x) = -x = -\mathrm{id}(x)$;
- The map $p_2 : S^1 \to S^1$ defined by $z \mapsto z^2$ is even, as $p_2(z) = z^2 = (-z)^2 = p_2(-z)$.
- The map $p_3 : S^1 \to S^1$ defined by $z \mapsto z^3$ is odd, as $p_3(-z) = -z^3 = -p_3(z)$.
- The (circular, hyperbolic) sine function is odd, while the (circular, hyperbolic) cosine function is even.
- The exponential function $\exp : \mathbb{R} \to \mathbb{R}$ is neither odd nor even.

Lemma 9.3. *The composition of,*

(i) Two even functions is even;

(ii) Two odd functions is odd;

(iii) An even and odd function (in either order) is even;

(iv) Any function with an even function is even (but not the reverse).

Proof. Let f be any function and suppose g is even. Then,

$$\begin{aligned}(f \circ g)(-x) &= f\big(g(-x)\big)\\ &= f\big(g(x)\big)\\ &= (f \circ g)(x)\end{aligned}$$

so $f \circ g$ is even. This covers (i), the reverse of (iii), and (iv).

For (ii), suppose f,g are odd. Then,

$$\begin{aligned}(f \circ g)(-x) &= f\big(g(-x)\big)\\ &= f\big(-g(x)\big)\\ &= -f\big(g(x)\big)\\ &= -(f \circ g)(x)\end{aligned}$$

and $f \circ g$ is odd.

For the other direction of (iii), suppose f is even and g is odd. Then,

$$\begin{aligned}(f \circ g)(-x) &= f\big(g(-x)\big)\\ &= f\big(-g(x)\big)\\ &= f\big(g(x)\big)\\ &= (f \circ g)(x)\end{aligned}$$

so $f \circ g$ is even. ■

9.2 Null-Homotopic Maps

A map $f : X \to Y$ is *null-homotopic* if it is free homotopic to a constant map. That is, if there exists a constant map e and a free homotopy $F : X \times I \to Y$ with $f_0 = f$ and $f_1 = e$.

A pointed map $f : (X,x_0) \to (Y,y_0)$ is *null-homotopic relative to the basepoint* if it is relatively homotopic to the constant map e_{y_0}. That is, there exists a homotopy $F : X \times I \to Y$ such that

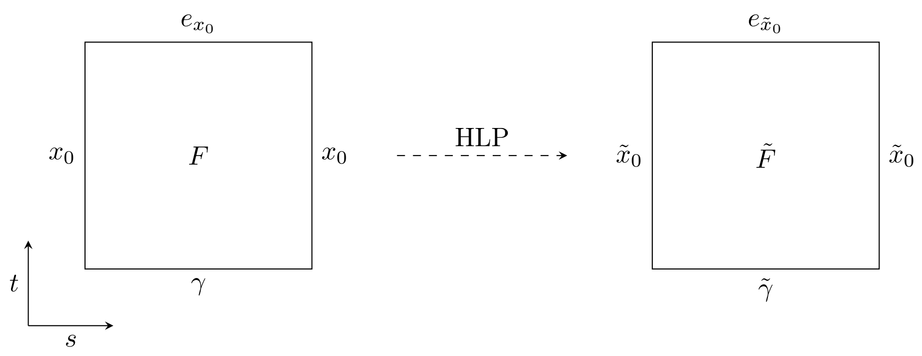
- $f_0 = f$;
- $f_1 = e_{y_0}$;
- $f_t(x_0) = y_0$ for all $t \in I$.

Conistent with the earlier notation for general relative homotopies, if f is null-homotopic relative to the basepoint x_0, we write $f \overset{x_0}{\simeq} e$.

Note that, even if the map $f : X \to Y$ is homotopic to e_{y_0}, and $x_0 \in X$ is such that $f(x_0) = y_0$, we do not necessarily have that the pointed map $f : (X,x_0) \to (Y,y_0)$ is null-homotopic, because a homotopy for the former need not preserve the pointedness of the intermediary maps f_t, while a null-homotopy of a pointed map further requires that $f_t(x_0) = y_0$ for all t.

Lemma 9.4. *Let $p : \tilde{X} \to X$ be a covering, and let $\gamma : I \to X$ be a loop such that $\gamma \overset{\partial}{\simeq} e_{x_0}$. Let $\tilde{x}_0 \in p^{-1}\big[\{x_0\}\big]$, and let $\tilde{\gamma}$ be the lift of γ with $\tilde{\gamma}(0) = \tilde{x}_0$. Then, $\tilde{\gamma} \overset{\partial}{\simeq} e_{\tilde{x}_0}$.*

Proof. Let $F : I \times I \to X$ be a homotopy between γ amd e_{x_0}. By the homotopy lifting property of coverings, there is a unique homotopy $\tilde{F} : I \times I \to \tilde{X}$ between $\tilde{\gamma}$ and $e_{\tilde{x}_0}$:



The left, right, and upper boundaries of the square on the right are all constant paths at x_0, so $\tilde{\gamma}$ is a loop at $\tilde{x}_0$, which is homotopic to $e_{\tilde{x}_0}$ via $\tilde{F}$. ■

Theorem 9.5. *If $f : S^1 \to S^1$ is odd, then f is not null-homotopic.*

Proof. WIP ■

Corollary 9.5.1. *If $f : S^2 \to \mathbb{R}^2$ is odd, then f has a root.*

Proof. WIP ■

9.2.1 The Borsuk-Ulam Theorem

Theorem (Borsuk-Ulam)**.** *For any continuous map $f : S^2 \to \mathbb{R}^2$, there exists a point $x \in S^2$ with $f(x) = f(-x)$.*

That is, for any continuous mapping of the sphere to $\mathbb{R}^2$, there exists two antipodal points for which the mapping has the same value.

One famous example of this theorem notes that mapping points on the Earth's surface to their temperature and atmospheric pressure can reasonably be assumed to be a continuous mapping, so the Borsuk-Ulam theorem states that at any time, there exist two antipodal points on the Earth's surface with equal temperature and atmospheric pressure.

Proof. Define $g : S^2 \to \mathbb{R}^2$ by
$$g(x) \coloneqq f(x) - f(-x)$$
We have $g(-x) = f(-x) - f(x) = -g(x)$, so g is odd, so by the previous corollary, g has a zero. That is, some $x \in S^2$ such that

$$\begin{aligned} g(x) &= 0 \\ f(x) - f(-x) &= 0 \\ f(x) &= f(-x) \end{aligned}$$ ■

9.3 Fundamental Groups of Product Spaces

Theorem 9.6. *Let (X,x_0) and (Y,y_0) be pointed spaces. Then,*

$$\pi_1(X \times Y, x_0 \times y_0) \cong \pi_1(X,x_0) \times \pi_1(Y,y_0)$$

That is, π_1 preserves binary products.

Proof. By the definition of the product topology, a map $Z \to X \times Y$ is continuous if and only if the components $p_1 \circ f$ and $p_2 \circ f$ are continuous, so a loop $\gamma : I \to X \times Y$ is equivalent to a pair of loops $\gamma_1 : I \to X$ and $\gamma_2 : I \to Y$.

Similarly, a homotopy F between loops in $X \times Y$ is equivalent to a pair of homotopies F_1 and F_2 between the equivalent loops in X and Y. That is, if $\alpha \overset{\partial}{\simeq} \beta$, then $p_1 \circ \alpha \overset{\partial}{\simeq} p_1 \circ \beta$, and $p_2 \circ \alpha \overset{\partial}{\simeq} p_2 \circ \beta$.

This induces a bijection $[\gamma] \mapsto \big([p_1 \circ \gamma],[p_2 \circ \gamma]\big)$, which gives the required isomorphism. ■

Example. The torus $T^2 = S^1 \times S^1$ with basepoint (1,1) has fundamental group

$$\pi_1\big(T^2,(1,1)\big) \cong \pi_1(S^1,1) \times \pi_1(S^1,1) \cong \mathbb{Z} \times \mathbb{Z}$$

Corollary 9.6.1. *By induction,*

$$\pi_1\left(\prod_{i=1}^{n}(X_i x_i)\right) = \prod_{i=1}^{n} \pi_1(X_i,x_i)$$

Example. The torus $T^n = \prod_{i=1}^{n} S^1$ has fundamental group

$$\pi_1(T^n) \cong \prod_{i=1}^{n} \pi_1(S^1) \cong \mathbb{Z}^n$$

Theorem 9.7. *For all $n \geq 2$, we have $\pi_1(S^n) \cong 0$.*

Proof. WIP ■

10 Galois Correspondence

Lemma 10.1. *Let $p : \tilde{X} \to X$ be a covering, and let $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}\big[\{x_0\}\big]$. Then,*

(i) The induced homomorphism $p_* : \pi_1(\tilde{X},\tilde{x}_0) \to \pi_1(X,x_0)$ is injective;

(ii) If $[\alpha] \in \pi_1(X,x_0)$, and $\tilde{\alpha}$ is the lift of α with $\tilde{\alpha}(0) = \tilde{x}_0$, then $\tilde{\alpha}$ is a loop if and only if $[\alpha] \in p_*\big(\pi_1(\tilde{X},\tilde{x}_0)\big)$.

Proof.

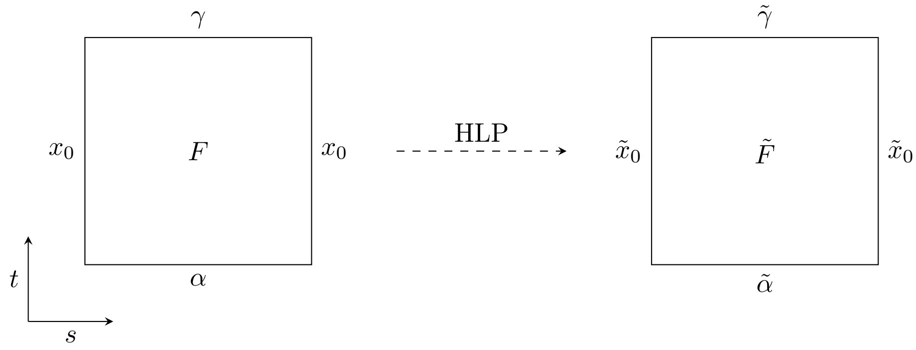
(i) Suppose p_* sends $[\tilde{\alpha}] \in \pi_1(\tilde{X},\tilde{x}_0)$ to the constant loop $[e_{x_0}]$. That is, $p \circ \tilde{\alpha} \overset{\partial}{\simeq} e_{x_0}$. Then by Lemma 9.4, $\tilde{\alpha} \overset{\partial}{\simeq} e_{\tilde{x}_0}$, so $[\tilde{\alpha}] = [e_{\tilde{x}_0}]$ and p_* has trivial kernel.

(ii) If $\tilde{\alpha}$ is a loop, then $p_*\big([\tilde{\alpha}]\big) = [p \circ \tilde{\alpha}] = [\alpha] \in p_*\big(\pi_1(\tilde{X},\tilde{x}_0)\big)$.

Conversely, suppose that $[\alpha] = p_*\big([\tilde{\gamma}]\big)$ for some $[\tilde{\gamma}] \in \pi_1(\tilde{X},\tilde{x}_0)$, so

$$\begin{aligned} \alpha &= p \circ \tilde{\alpha} \\ &\overset{\partial}{\simeq} p \circ \tilde{\gamma} \\ &= \gamma \end{aligned}$$

so there is some relative homotopy F from α to γ that lifts to a homotopy from $\tilde{\alpha}$ to $\tilde{\gamma}$:



The left and right boundaries of F are constant, so they lift to constant paths at $\tilde{x}_0$, so $\tilde{\alpha}$ is a loop (based at $\tilde{x}_0$) as required.

■

This shows that for any covering p, the image $p_*\big(\pi_1(\tilde{X},\tilde{x}_0)\big)$ is a subgroup of $\pi_1(X,x_0)$ that is isomorphic to $\pi_1(\tilde{X},\tilde{x}_0)$.

Example. The covering $p_2 : S^1 \to S^1$ induces the doubling map $n \mapsto 2n$, so

$$(p_2)_*\big(\pi_1(S^1,1)\big) \cong 2\mathbb{Z} \leq \mathbb{Z} \cong \pi_1(S^1,1)$$

Let $p : \tilde{X} \to X$ be a covering, and suppose X is connected. Then, the cardinality of the preimage of any point in X is called the *degree* of the covering:

$$\deg(p) := \big|p^{-1}\big[\{x\}\big]\big|$$

Recall that, given a group G and a subgroup $H \leq G$, the *index* $[G : H]$ of H in G is the number of right (or left) cosets $G/H = \{Hg : g \in G\}$.

Lemma 10.2. *Let $p : \tilde{X} \to X$ be a covering and suppose that $\tilde{X}$ and X are path-connected. Let $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}\big[\{x_0\}\big]$. Then,*

$$\deg(p) = \Big[\pi_1(X,x_0) : p_*\big(\pi_1(\tilde{X},\tilde{x}_0)\big)\Big]$$

Proof. WIP ■

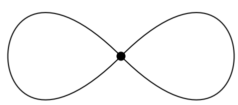
11 Wedge Sums

Let $\big\{(X_\alpha,x_\alpha)\big\}_{\alpha\in\Lambda}$ be a collection of pointed spaces. The *wedge sum* of this collection is the "one-point union" of the spaces, defined as:

$$\bigvee_{\alpha\in\Lambda}(X_\alpha,x_\alpha) := \bigsqcup_{\alpha\in\Lambda} X_\alpha/\sim x_\beta$$

That is, the disjoint union of each space with all the basepoints identified.

Example. The wedge sum of two pointed circles is the figure-eight graph:



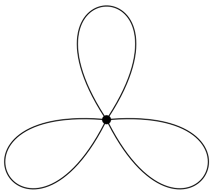
$S^1 \vee S^1$

The identified point is a natural choice of basepoint for the wedge sum, and selecting this point makes the wedge sum associative and commutative (up to homeomorphism) over pointed spaces, as every basepoint is always identified to the same point, so associativity and commutativity follow from disjoint unions being associative and commutative.

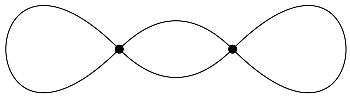
However, we may also treat the output as an ordinary topological space without any distinguished basepoint, in which case, the wedge sum is then *not* associative, as a new basepoint may be selected between applications.

If an expression involving wedge sums is not bracketed, we will assume that the natural basepoint is selected, so the resulting space is unambiguous and unique.

Example.



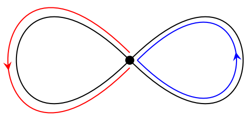
$S^1 \vee S^1 \vee S^1$



$(S^1 \vee S^1) \vee S^1$

In the first case, all three basepoints coincide, resulting in the bouquet of three circles. In the second, we were free to pick a basepoint distinct from the centre of the figure-eight, where the third circle was adjoined.

Let us mark two loops a and b on $S^1 \vee S^1$:



$S^1 \vee S^1$

and denote their reverse paths by a^{-1} and b^{-1}.

Note that any loop in $S^1 \vee S^1$ can be decomposed into a string consisting of the symbols a, b, a^{-1}, and b^{-1}. For example,

$$aaba^{-1}b^{-1}a$$

corresponds to the loop that travels along a twice, b once, a backwards, b backwards, then a.

Some strings of this form may be *reduced* up to homotopy, as any substring consisting of a loop adjacent to its inverse is homotopic to the constant loop, which may then be removed from the string.

This structure is well-suited to be described by *free products*.

11.1 The Free Product of Groups

Let $\{G_\alpha\}_\alpha$ be a collection of groups. A *word* on these groups is a finite sequence $g_1 \cdots g_m$ of elements $g_i \in G_{\alpha_i}$, and m is the *length* of the word. The empty word of length 0 is denoted by ε. The *product* of two words is their concatenation

$$(g_1 \cdots g_m) * (h_1 \cdots h_n) = g_1 \cdots g_m h_1 \cdots h_n$$

A word is *reduced* if it does not contain the identity of any group, and if every pair of consecutive letters is not from the same group.

Given any word g on the groups $\{G_\alpha\}_\alpha$, we can *reduce* it to a reduced word g' by recursively removing all identity elements and replacing any consecutive elements g_i, g_{i+1} from the same group with their group product $g_i \cdot g_{i+1}$.

Let $*_\alpha G_\alpha$ be the set of reduced words on $\{G_\alpha\}_\alpha$. We can define an operation on this set as follows: given reduced words $g = g_1 \cdots g_m$ and $h = h_1 \ldots h_n$, construct a new reduced word $g \bullet h$ by taking the concatenation $g * h$, then reduce the word recursively

$$g \bullet h = \begin{cases} gh & g_m \in G_\alpha, h_1 \in G_\beta,\, G_\alpha \neq G_\beta \\ g_1 \cdots g_{m-1}(g_m \cdot h_1)h_2 \cdots h_n & g_m, h_1 \in G_\alpha,\, g_m \cdot h_1 \neq \mathrm{id}_{G_\alpha} \\ g_1 \cdots g_{m-1} \bullet h_1 \cdots h_n & g_m, h_1 \in G_\alpha,\, g_m \cdot h_1 = \mathrm{id}_{G_\alpha} \end{cases}$$

Then, $(*_\alpha G_\alpha, \bullet)$ is a group called the *free product* of $\{G_\alpha\}_\alpha$, with identity ε, and the inverse of an element $g_1 \cdots g_m$ is given by $g_m^{-1} \cdots g_1^{-1}$.

Example. The free product of $\mathbb{Z}_2$ with itself is given by the semidirect product $\mathbb{Z}_2 * \mathbb{Z}_2 \cong \mathbb{Z} \rtimes \mathbb{Z}_2$.

Example. If $G = \langle a \mid a^4 \rangle$ and $H = \langle b \mid b^5 \rangle$, then $G * H = \langle a, b \mid a^4 = b^5 \rangle$.

Every group G_α is a subgroup of the free product $*_\alpha G_\alpha$ via the inclusion $\iota_\alpha : G_\alpha \hookrightarrow *_\alpha G_\alpha$ that maps each non-identity $g \in G_\alpha$ to the string g, and the identity to the empty string. The free product satisfies the following universal property:

Lemma 11.1. *Any pair of homomorphisms from groups G and H into K factor uniquely through the free product.*

That is, for any group homomorphisms $\varphi : G \to K$ and $\psi : H \to K$, there exists a unique homomorphism $\varphi * \psi : G * H \to K$ such that

$$\begin{tikzcd} G \arrow[r, hook, "\iota_1"] \arrow[dr, "\varphi"'] & G * H \arrow[d, dashed, "\varphi * \psi"] & H \arrow[l, hook', "\iota_2"'] \arrow[dl, "\psi"] \\ & K & \end{tikzcd}$$

commutes.

This holds more generally, with a collection of homomorphisms $\varphi_\alpha : G_\alpha \to K$ factoring uniquely through a map $*_\alpha \phi_\alpha : *_\alpha G_\alpha \to K$:

$$\begin{tikzcd} G_\alpha \arrow[r, hook, "\iota_\alpha"] \arrow[dr, "\varphi_\alpha"'] & *_\alpha G_\alpha \arrow[d, dashed, "*_\alpha \varphi_\alpha"] \\ & K \end{tikzcd}$$

12 The Seifert-van Kampen Theorem

Let X be a topological space and $\{U_\alpha\}_\alpha$ be an open cover with inclusion maps $\iota_\alpha : U_\alpha \hookrightarrow X$, and further suppose that the intersection $\bigcap_\alpha U_\alpha$ is non-empty.

Let $x_0 \in \bigcap_\alpha U_\alpha$, and consider the pointed spaces (U_α, x_0). The inclusion maps $\iota_\alpha : U_\alpha \to X$ induce homomorphisms between the fundamental groups based at x_0:

$$(\iota_\alpha)_* : \pi_1(U_\alpha, x_0) \to \pi_1(X, x_0)$$

which factor through the free product map

$$\Phi = *_\alpha(\iota_\alpha)_* : *_\alpha\pi_1(U_\alpha,x_0) \to \pi_1(X,x_0)$$

If the pairwise intersections $U_a \cap U_b$ are path-connected, then Φ is surjective; but in general, it is not injective, as loops in the intersections are counted twice in the free product.

The inclusions $\iota_{ab} : U_a \cap U_b \to U_a$ of the intersections then also induce maps between fundamental groups, completing the commutative diagram:

$$\begin{tikzcd}
 & \pi_1(U_a,x_0) \arrow[dr, hook, "i_a"] \arrow[drr, bend left, "(\iota_a)_*"] & & \\
\pi_1(U_a \cap U_b,x_0) \arrow[ur, hook, "(\iota_{ba})_*"] \arrow[dr, hook, "(\iota_{ba})_*"'] & & *_\alpha\pi_1(U_\alpha,x_0) \arrow[r, dashed, "\Phi"] & \pi_1(X,x_0) \\
 & \pi_1(U_b,x_0) \arrow[ur, hook, "i_b"'] \arrow[urr, bend right, "(\iota_b)_*"'] & &
\end{tikzcd}$$

In categorical language, $(\iota_a)_*$ and $(\iota_b)_*$ form a pushout for all a,b.

Now, note that every class $\omega \in \pi_1(U_a \cap U_b,x_0)$ is represented twice in $*_\alpha\pi_1(U_\alpha,x_0)$ as $(\iota_{ab})_*(\omega)$ and as $(\iota_{ba})_*(\omega)$. Define the set

$$V_{ab} = \left\{(i_{ab})_*(\omega)(i_{ba})_*(\omega)^{-1} : \omega \in \pi_1(U_a \cap U_b,x_0)\right\}$$

and define $V = \bigcup_{a,b} V_{ab}$. We then define N to be the normal closure of V. That is, the minimal normal subgroup N of $*_\alpha\pi_1(U_\alpha,x_0)$.

Theorem (Seifert-van Kampen)**.** *Let X be a topological space, $\{U_\alpha\}_\alpha$ be an open cover with non-empty intersection, and x_0 some point in $\bigcap_\alpha U_\alpha$. Then,*

(i) If the intersection $U_a \cap U_b$ is path-connected for all a,b, then the free product map

$$\Phi = *_\alpha(\iota_\alpha)_* : *_\alpha\pi_1(U_\alpha,x_0)$$

is surjective.

(ii) If in addition the intersection $U_a \cap U_b \cap U_c$ is path-connected for all a,b,c, then $\ker(\Phi) = N$ *and hence*

$$\pi_1(X,x_0) \cong *_\alpha\pi_1(U_\alpha,x_0)\big/N$$

Example. Consider the sphere S^n for $n \geq 2$, with the cover $\{U_1,U_2\}$ given by the sets obtained by deleting two distinct points from the sphere. The intersection is path-connected, so

$$\Phi : \pi_1(U_1,x_0) * \pi_1(U_2,x_0) \to \pi_1(S^n,x_0)$$

is surjective. The open sets U_1 and U_2 are also both homeomorphic to $\mathbb{R}^n$, which is contractible, so their fundamental groups, and hence the free product, are trivial, so $\pi_1(S^n,x_0)$ must also be trivial.

This argument fails for S^1 as the intersection $U_1 \cap U_2$ is disconnected.

Example. Let $(X,x) = \bigvee_\alpha(X_\alpha,x_\alpha)$ be the wedge product with natural basepoint $x = [x_\alpha]$, and suppose that for every α, there exists a contractible open neighbourhood $N_\alpha \subseteq X_\alpha$ of x_α. Then, for each α, define $U_\alpha = X_\alpha \vee \bigvee_{\beta\neq\alpha} N_\beta$.

Each U_α is open in X, and the basepoint is contained in their intersection $\bigcup_\alpha U_\alpha$ by construction, and each pairwise intersection is $\bigvee_\alpha U_\alpha$, which deformation retracts to x, i.e. is contractible. It follows that the pairwise intersections have trivial fundamental groups, so by Seifert-van Kampen, we have

$$\pi_1\left(\bigvee_\alpha X_\alpha, x\right) \cong *_\alpha \pi_1(X_\alpha, x_\alpha)$$

13 Generators and Relations

A *presentation* is a method of specifying a group G via a set S of *generators* – so that every element of the group may be expressed as a product of generators – and a set R of *relations* between those generators, and we write that G has presentation

$$\langle S \mid R\rangle$$

Informally, G is the "most general" or "freest" group generated by S constrained only by relations in R. Formally, G has presentation $\langle S \mid R\rangle$ if it is isomorphic to

$$G \cong \langle S\rangle / \langle\langle R\rangle\rangle$$

where $\langle\langle R\rangle\rangle$ is the normal subgroup generated by R.

Example. The cyclic subgroup $\mathbb{Z}_n$ has presentation

$$\langle a \mid a^n = 1\rangle$$

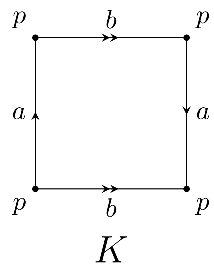
This may also be written as

$$\langle a \mid a^n\rangle$$

where the convention is that any terms without an equality symbol are taken to be equal to the group identity.

A group is *finitely generated* if its set of generators S is finite; *finitely related* if its set of relators R is finite; and *finitely presented* if both S amd R are finite.

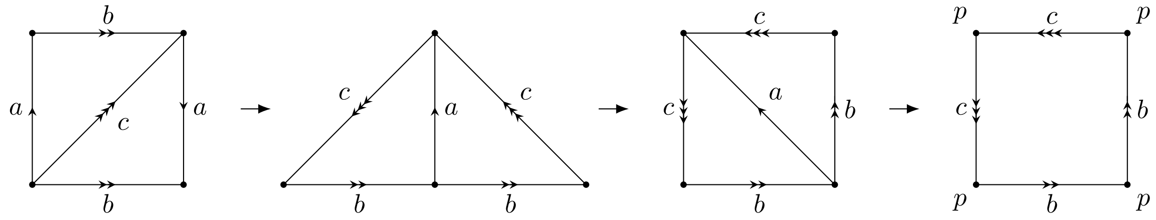
Example. Consider the Klein bottle K,



The 1-skeleton X^1 consists of the loops a and b, and the 0-skeleton is the single point p, so the generators are the classes corresponding to the cycles a and b, and the single relation is the loop that forms the boundary, given by $baba^{-1}$, so we have the presentation

$$\langle a, b \mid baba^{-1}\rangle$$

However, there are several cell structures on K. One rearrangement is as follows:



The resulting presentation is then

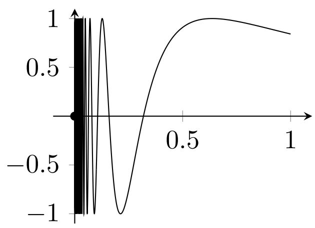
$$\langle b,c \mid b^2c^2 \rangle$$

This is of course obtainable purely group-theoretically by defining new elements in terms of old ones, but we also see that each presentation corresponds to a different way of describing a topological space.

14 List of Useful (Counter)examples

- The *topologist's sine curve* is the subspace $T \subseteq \mathbb{R}^2$ defined by

$$T = \left\{ \left(x, \sin \frac{1}{x} \right) : x \in (0,1] \right\} \cup \{(0,0)\}$$

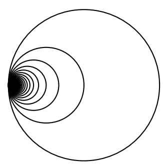


  The topologist's sine curve is connected but not path-connected because the origin cannot be separated from the rest of the curve, but also cannot be connected to the rest of the curve via a path.

- Define $C_n \subseteq \mathbb{R}^2$ as the circle of radius $1/n$ centred at $(0,1/n)$. The *Hawaiian earring* is the union

$$H = \bigcup_{n \in \mathbb{Z}^+} C_n$$

  equipped with the subspace topology.



  The Hawaiian earring looks similar to the infinite wedge sum

$$X = \bigvee_{n \in \mathbb{N}} S^1$$

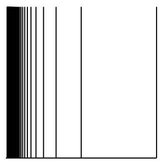
  but they are not homeomorphic:

  - The fundamental group $\pi_1(X)$ is countable, while $\pi_1(H)$ is uncountable.
  - The Hawaiian earring is compact, while the wedge sum is not.

  - In the Hawaiian earring, every open neighbourhood of the intersection point completely contains all but finitely many of the circles (i.e. an ε-ball around (0,0) contains every circle whose radius is less than $\varepsilon/2$), while in the wedge sum, such a neighbourhood may contain no circles at all.

- The *topologist's comb* is the subset $C \subseteq \mathbb{R}^2$ defined by

$$C = \big(\{0\} \times [0,1]\big) \cup \left(\left\{\frac{1}{n} : n \in \mathbb{N}^+\right\} \times [0,1]\right) \cup \big([0,1] \times \{0\}\big)$$



  The comb space is contractible but does not deformation reformation to any point on the line segment $\{0\} \times [0,1]$.

- The *closed long ray* is the product of the first uncountable ordinal ω_1 with the half-open interval $[0,1)$:

$$L = \prod_{i \in \omega_1} [0,1)$$

  equipped with the lexicographical order topology. (Compare with the real number line, which can be constructed as the product of $\mathbb{N}$ copies of $[0,1)$.)

  The *open long ray* is obtained by removing (0,0), and the *long line* is then obtained by gluing together two copies of the closed long ray at the origin.

  - The long rays and line are path-connected but not contractible.
  - The long rays and line are normal, Hausdorff, and sequentially compact, but not compact nor metrisable.